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This Journal is dedicated to the following aims:

1. Through published standard papers on the culture aspects, humanism and history of mathematics to deepen and to widen public interest in its values.
2. To supply an additional medium for the publication of expository mathematical articles.
3. To promote more scientific methods of teaching mathematics.
4. To publish and to distribute to groups most interested high-class papers of research quality representing all mathematical fields.

To Have and to Hold

The need for mathematics in daily affairs and the need for its disciplinary mental training have been so strongly stressed upon the child that in entering upon his educational life he accepts it much in the same way that we all submit to the common cold. Coming into the class, he seats himself and says, "Well, here I am, try to teach *me*". Who of us has not despaired of all hope at that impregnable stone wall of passive resistance? But when he grows up and becomes a member of the board of education that throws mathematics overboard, who's to blame?

It seems, unfortunately, that too many of us have lost all pretense of making mathematics vital and interesting. But vitalize it we must. No longer may we hold the attitude that "it is a prime necessity of life—here it is—come and get it". We must fight first and last for the interest of that youngster who sits there impolitely gazing out of the window. Prick his interest sharply and he's yours. But fail to do that and the blame for that action of the board will lie squarely in your own lap.

It takes but little sometimes to start that driving mechanism which we call interest. In high school geometry where the student is most likely to feel overburdened with purely mental speculation, let him try to solve some of the problems of Euclid by means of a parallel ruler instead of the classical tools. Or restrict him to the compasses alone. Or let him cut a pie-shaped piece of wood to serve as his only instrument. That bright morning when he's first introduced to the theorem of Pythagoras, ask him to follow you into the wood shop. Imagine his curiosity. If he makes two wood squares and then saws them into pieces that will fit together to make a third square he'll not only possess an explanation but will consider the theorem almost his very own. Jig saw puzzles instead of drudgery. Let him linger around and saw up a quadrilateral into pieces that will make a triangle. Holding mathematics in his hand for a change will freshen him more than dew does the morning glory.

Give him a sheet of paper on which three points are marked. Then ask him to find the circumcenter of that triangle, the orthocenter, the centroid, the incenter, etc.—entirely without tools—even a pencil. He'll think you've gone a bit balmy at first but soon he'll

discover that beautiful art of paper folding which, sadly enough, seems to have been almost lost in the rush of present things.

Let him construct in this mathematical laboratory a cone of light and discover for himself those marvelous curves, the ellipse, parabola, and hyperbola. Or, in the Gilbert and Sullivan vein, build a miniature elliptical billiard table so that, blindfolded, he may shoot the cue ball at random from one focus and never miss the eight ball at the other. A bucket of wet plaster of paris attached to a long and twisted vertical cord will give a nice parabolic cast which can be shellacked and polished for a mirror. A camera time exposure at night of the shots from a roman candle will record evidence of near parabolic paths and their parabolic envelope.

Let him play with soap bubbles and wires and gelatin solutions of various colors. Some of these solutions can be made up to give fairly permanent films that can be put on display. Who knows but here might be laid the seed of intimacy with future work on minimal surfaces? Show him the difference in form between a chain freely suspended and one that supports a uniform horizontal load. Ask him if he can think of any point that actually travels backwards on a locomotive streaking ahead at sixty miles per hour. Then give him a wheel with an extended radial arm to roll on the chalk ledge and trace a prolate cycloid. Let him construct with a light flexible wooden strip a cycloidal track down which a ball may roll and compare its time with that for a straight strip.

Introduce him to the epi- and hypocycloids by presenting him with circular disks and rings of various sizes. Ask him to fix up an apparatus to give the hypocycloid of four cusps, of three, and finally of two. He'll there discover line motion. Let him experiment with caustic curves by using brightly polished rings and light sources. Guide him into the discovery of the cardioid and then get him to produce the same curve with equal sized disks. Or with two rods and a slide in the manner of Pascal.

An aluminum alloy may be had that is light and strong enough to make side and end pieces for the quadrics, which may be strung with colored silk fish line. If a lathe is available, he would consider it a pleasure to turn out a wooden cone from which sections might be cut.

Ask him to make one of the many linkages that will cause a point to move freely along a line. The Peaucellier cell, the father of them all, is easy to construct with strips of cardboard or wood and will undoubtedly create a strong desire for information on others. Tell him that it was used to ventilate the House of Parliament and that Sylvester even attached it to his plumbing. Let him enjoy the pantograph and the skew pantograph. Give him the parallel motion used long ago by James Watt in his steam engine and even today on river boats and

motion picture projectors. Let him see that remarkable triple generation theorem inherent in the three bar curve. And some of the beauties of the Hart contraparallelogram. A score or more of these linkages require but simple methods of plane geometry to establish their principles.

Let him trisect angles with the carpenter's square, or the marked ruler, the two footed compasses of Hermes, the tomahawk, Sylvester's Lady's Fan, or a half dozen other gadgets. Or use these instruments to construct some of the "outlawed" regular polygons.

A funnel suspended by strings from the ceiling, when allowed to swing back and forth, will trace in sand those curves we find useful in mechanics. If the ceiling is high, drop a small ball by a cord directly over and touching a cone. Then let him try to swing the ball freely past the cone so that it might be struck on the return. He is doomed to probable failure, of course, since the path is approximately an ellipse with axes that are different from zero.

If space is limited or too valuable in school classrooms for this laboratory try a corner of the janitor's hideaway. Or petition a good citizen for one side of his garage. But someway, somehow, start the ball rolling. Give that fellow a chance to wake up and live—to hold and fondle his mathematics and make it his own.

The county fair? Indeed yes—but isn't it about time?

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Relativity*

By B. Hoffmann
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1. *Introduction.* The mathematics of relativity is on the whole quite simple, the much advertised difficulties of the theory residing mainly in its physical implications, which run counter to some of our most cherished everyday notions. A first glance at the theory of relativity is invariably disappointing. One naturally expects an admittedly important theory to concern itself with admittedly important matters. Relativity, however, talks endlessly of *space* and *time*, and *simultaneity* and other familiar things. One forgets all too easily that such familiar things are of the first importance for physical science; that they are part of that shaky foundation on which is balanced the whole intricate and beautiful structure of scientific theory and philosophical thought. To tamper with them is to send a shudder coursing from one end of this vast frame to the other. And to effect successfully a profound change in our ideas concerning them, as Einstein did, is to create a revolution in science and philosophy of transcendental importance.

In a short article such as this, much must be sacrificed for the sake of brevity, and rather than be general and vague I have tried to present a reasonably specific account of one particular part of the subject; the so-called special theory of relativity. Admittedly the general theory, bristling with tensors set in a Riemannian space, has greater purely mathematical interest, but the simplest approach to Einstein's revolutionary physical ideas is undoubtedly through the special theory.

2. *On Newtonian Dynamics.* Let us begin with the Newtonian laws of motion. Not the law of gravitation. That comes in much later. Simply the three laws of motion. These laws are far too well known to warrant repetition here. They may be found in the original Latin in any reasonably complete edition of Newton's *Principia*, and are given in elegant English prose in every text-book on dynamics. For our present purposes, let it suffice to say that they lead, with the usual notation and significance, to the equations.

$$F_x = \frac{d}{dt}\left(m\frac{dx}{dt}\right), \tag{1}$$

*This is the second of a series of expository articles written at the invitation of the Editors.

$$F_y = \frac{d}{dt}\left(m \frac{dy}{dt} \right),$$

$$F_z = \frac{d}{dt}\left(m \frac{dz}{dt} \right).$$

However, it may not suffice long. We have important questions to ask concerning these equations, questions concerning their conceptual background. For instance, precisely what are x, y, and z in (1)? And what is t ? What do we mean by the mass m ? Is the force (F_x,F_y,F_z) something which produces a change of momentum in accordance with (1), or is it perhaps just a convenient shorthand for the vector whose components are the right-hand sides in (1)? These and similar questions.

Newton of course gave careful thought to such matters. Space—simple (x,y,z)—presented little difficulty, for Euclid and Descartes had done their work all too well. Newton adopted what they had developed, and space for him was an unbounded Euclidean nothingness, without form and void, and untouched by the many wondrous things within it. We shall see a quite different concept arising out of Einstein's researches.

Time presented a more difficult problem, as you may readily convince yourself. Just look it up in the dictionary and see if you would care to construct a precise dynamical theory on what you find. In Newton's theory, time, like space, was untouched by the grosser things of the world, and remained aloof from all influence from material things, and even from space itself. One must mention these obvious things here since they are no longer regarded as true. Incidentally, in order to give time a really Olympian status, Newton asserted that it "flows uniformly", though he knew full well that such a statement is meaningless inasmuch as only time itself can time itself.

Of mass, Newton said it measured the quantity of matter in a body, and this definition, for all its vagueness, suffices for the theory of relativity, though with surprising modifications of meaning.

As to the Newtonian force, it must be admitted that the "shorthand" interpretation mentioned above is undoubtedly as good as any, despite our anthropomorphic feelings to the contrary. In the light of the general theory of relativity this interpretation remains the only reasonable one.

Now that all this has been mentioned for the record, let us investigate the behavior of the Newtonian equations of motion (1) under various transformations of coordinates. First let us change from the

original coordinates (x,y,z) to new coordinates (x',y',z') moving with constant velocity v relative to the first. For simplicity we assume the axes initially coincident, and their relative velocity along the common x-direction. From the definition of relative velocity, limit of relative displacement divided by time taken, it follows that after time t the origins will have separated by a distance vt. The law of transformation at time t is therefore easily seen to be

$$x=x'+vt, \quad y=y', \quad z=z'. \tag{2}$$

Though this is "easily seen", let us not lose sight of the fact that these transformation equations depend entirely on the Newtonian concepts of space and time. Neither the concepts nor the equations will remain unscathed as we proceed; nor, unhappily, will the relativistic equations of transformation be so "easily seen" as were these that are soon to be discarded.

For the present let us mention that one ought perhaps to supplement (2) with

$$t=t', \tag{2a}$$

but that to do this in Newton's theory would be to commit a redundance of the most pedantic type.

Under the transformation (2), it is (once more) easily seen that the equations of motion remain unaffected. The moral in this instance is that this is not true unless the force is unaffected by the relative motion. This is a tremendous assumption, of course. In fact all our experience with forces seems against it. We find it much easier to grip a stationary body than one that is rushing past us, yet we are here assuming that a force—say gravitational—can *grip* a moving body as easily as if it were stationary. It may be a customary assumption, but it is surely not an obvious one. Moreover it is definitely untrue of electro-magnetic forces, as any physics text book will tell.

The next step brings further revelations. We investigate the case in which the relative velocity is variable. It is enough to assume a constant acceleration a. The law of transformation at time t is now

$$x=x'+vt+\tfrac{1}{2}at^2, \quad y=y', \quad z=z' \tag{3}$$

and the Newtonian equations of motion become

$$F_x=\frac{d}{dt}\left(m\frac{dx'}{dt}\right)+ma, \tag{4}$$

$$F_y=\frac{d}{dt}\left(m\frac{dy'}{dt}\right),$$

$$F_z = \frac{d}{dt}\left[m \frac{dz'}{dt} \right].$$

The presence of the *ma*-term shows that Newton's equations—and therefore Newton's laws of motion—are invalid in the new system of coordinates. This accords with our everyday experience. The first law of motion, for example, is obviously untrue relative to an accelerating train.

Naturally Newton was well aware of this. Coordinate systems in which his laws are valid he called inertial systems. If the laws are valid in some particular coordinate system, they will be valid for all systems moving with constant velocity relative to it, and for no others. Thus all inertial frames have constant velocity relative to one another, and no experiment according to Newtonian dynamics can distinguish between them. Newtonian dynamics, in this sense, embodies that very ancient principle which asserts that all motion is relative; technically, that all inertial systems are equivalent for the description of the laws of nature. This is the important principle of relativity, and we shall follow its fortunes through a surprising adventure.

Before we leave this discussion of Newtonian dynamics we should mention a well-known trick which artificially saves the validity of the equations of motion for accelerated systems. All unwanted terms on the right—in (4) the term *ma*—are shifted bodily to the left with changed signs, and being now on the left, are referred to as fictitious forces introduced by the accelerations. Centrifugal forces due to rotations are a familiar example. Such forces can be just as powerful and destructive as the non-fictitious variety, and are "fictitious" only in the sense that they are best attributed to accelerations. The situation in Newtonian dynamics in this respect is quite unsatisfactory, and cannot be clarified until we reach Einstein's general theory of relativity.

Incidentally, another dubious matter of analogous significance is the role of time measurement in the definition of an inertial system. For if one's clock were slowly running down, then all free motions would seem to be steadily accelerated, and an inertial system would seem unrealizable. Thus time plays an important, even though little mentioned, part in the definition of inertial systems.

3. *The Paradox.* The successful development of the wave theory of light brought with it an important implication; the existence of an "ether" pervading all space. This ether could be used to define that elusive bird absolute rest. Motion relative to the ether would then be absolute motion, and the principle of relativity would be contradicted.

Naturally, there was no reason why the principle of relativity should be above contradiction. That it was valid for Newton's equations meant only that absolute motion could not be detected by purely mechanical experiments. It did not mean that optical experiments would fare no better.

It was Maxwell who suggested that the absolute velocity of the earth—its velocity relative to the ether—could be detected by measuring the difference in the speed of light in perpendicular directions. He believed such an experiment far too delicate ever to have a chance of being performed. But he reckoned without the ingenuity of Michelson who, by his invention of the interferometer, brought an immense increase in sensitivity and precision to the armory of experimental physics. In 1887 Michelson and Morley performed their historic ether drift experiment along the lines suggested by Maxwell. To their amazement they found no difference at all in the speed of light in different directions; in other words, no motion relative to the ether. This negative result, obtained repeatedly, was in flat contradiction with the expectations of contemporary science. Not only did it actually support the somewhat discredited principle of relativity, but, far worse, it established that the measured speed of light is independent of the velocity of the experimenter through the ether.

This is devastating. Imagine a lamp L and two observers, O and O'. Let O remain at a fixed distance from L while O' moves towards L with velocity v. Then, evidently, if O measures the speed of the light waves coming from L to be c, O' must obtain the value $(c+v)$. Yet, according to the incontrovertible evidence of the Michelson-Morley experiment, he will in actual practice obtain simply the value c.

As W. S. Gilbert once remarked under rather different circumstances* which, however, also had to do with a question of time, we have here

> "*A paradox, a paradox,*
> *A most ingenious paradox*!"

Yet, as he said on another occasion,† you may

> "*Search in and out and round about*
> *And you'll discover never*
> *A tale so free from every doubt—*
> *All probable, possible shadow of doubt—*
> *All possible doubt whatever*!"

**The Pirates of Penzance.*
†*The Gondoliers.*

4. *The Special Theory of Relativity.* In 1905 Einstein gave the resolution of the paradox in a remarkable paper *On the Electrodynamics of Moving Bodies*, propounding therein what we now call the special theory of relativity.

In accordance with the evidence of the Michelson-Morley experiment, Einstein took as his basic assumptions essentially the following principles:

I. *The Principle of Relativity: all systems having constant velocity relative to one another are equivalent for the expression of physical laws;*

II. *The speed of light is the same measured relative to all such systems.*

Now we have just seen that these lead to a serious contradiction. How then can we hope to build a consistent theory upon them? Einstein showed that the roots of the contradiction lay elsewhere; in our erroneous ideas of space and time, especially time. With inexorable logic he then extracted from his assumptions a new sort of space and time destined to have far reaching influence on physical theory.

The first step in the argument is to destroy the usual notion of simultaneity at different places. This has to be done in face of enormous psychological resistances, but after the point is conceded everything else is comparatively smooth sailing. We begin very carefully by imagining a coordinate system S thickly studded with accurate clocks. If these clocks are exactly synchronous, it is a simple matter to determine whether two events at different places are simultaneous or not. If, for example, one event occurs when the clock nearest it says noon, and the other event occurs when the clock nearest *it* says noon, we may justifiably consider the two events simultaneous. Everything hinges now upon the synchronization of the multitudinous clocks. How are we to acheive this synchronization with theoretically infinite precision? Will it do to set all the clocks according to a master clock carried from place to place? This is not reliable, for acceleration of the master clock would jar its mechanism and thus destroy its value as a synchronizer.*

To be on the safe side, Einstein proposed using II as the basis of synchronization. Consider two clocks A and B fixed at different places in the coordinate system S. Let light be sent from A when A reads t_a; let it reach B when B reads t_b; and let it be immediately reflected back

*Even uniform motion of clocks is suspect in view of the Michelson-Morley experiment, which led Fitzgerald to suggest, and Lorentz to amplify, the idea that objects shrink when moving through the ether. A shrunken, elliptic-looking clock is certainly not to be trusted!

to A, reaching it when A reads t_a'. According to II, the time taken for the forward journey must be equal to the time taken for the trip back. We must therefore define the clocks A and B to be synchronous if and only if

$$t_b - t_a = t_a' - t_b. \tag{5}$$

It is further to be assumed that every other correct method of defining synchronization must be esssentially equivalent to this. The definition is a most reasonable one, and indeed all goes well so long as we remain in one coordinate system.

But observe what happens when we compare coordinate systems in relative motion. We need merely consider how the above synchronization looks to an observer using a coordinate system moving with velocity v relative to S.

In S the path of the light looked like this

and the times shown by the clocks A and B are such that the forward and backward journeys register equal intervals of time.

In S', however, assuming it to be moving to the right relative to S, the path looks like this

According to II, the speed of light must be c for S' as well as for S. Thus S' will maintain that the forward journey took less time than the journey back, and will therefore vehemently deny that clocks A and B are synchronous. In view of I, it necessarily happens that S must spread identical gossip about the clocks of S'. Simultaneity at different places is thus seen to be a relative concept, depending on the motion of the coordinate system for which it is defined. Classical physics did not know of II, and consequently saw no difficulty in the above situation. It preserved absolute simultaneity by allowing speeds $(c-v)$ and $(c+v)$ for the light sent out by S as viewed by S'. Einstein accepted II, and in consequence was forced to reject the concept of absolute simultaneity.

Rather than proceed pictorially as above, let us now obtain all the other peculiar results of the theory of relativity that we wish to discuss as mathematical deductions from the law of transformation between S and S' required by the basic postulates. The argument on simultaneity will have served its purpose if the initial assumption that

we now make appears reasonable, namely that a coordinate system must be considered as being studded with synchronized clocks, and that t must represent the time shown by the clock at the general point (x,y,z), and not simply "the time" in the absolute sense of classical physics. We must regard t as intimately related to (x,y,z) and must introduce a t' for S' similarly related to (x',y',z').

To simplify matters we assume S and S' initially coincident, with the clocks at the respective origins initially registering zero. We further assume S' to be moving with uniform velocity v relative to S along the common x-axis, this velocity being measured by S.* We shall leave z and z' out of our considerations. And finally, we shall introduce the quantity

$$\xi \equiv x - vt, \tag{6}$$

since a point fixed in S' has a ξ independent of t.

With the preliminaries all settled, we begin by assuming a linear relation with constant coefficients between $(x,y;t)$ and $(x',y';t')$, which implies a like relation between $(\xi,y;t)$ and $(x',y';t')$. So we write

$$\begin{aligned} x' &= l_1\xi + m_1y + n_1t, \\ y' &= l_2\xi + m_2y + n_2t, \\ t' &= l_3\xi + m_3y + n_3t. \end{aligned} \tag{7}$$

Now consider the synchronization by S' of a clock on his x'-axis with the clock at his origin O', this synchronization being performed in accordance with the recipe previously given. Let the clock have the general position $(x',0)$, and let its position according to S be given by $(\xi,0)$. Suppose that the light leaves O' at $t'=0$, reaches x' at $t'=t_1'$, and returns to O' at $t'=t_2'$. Then, by (5), S' must have

$$t_1' - 0 = t_2' - t_1',$$

or

$$2t_1' = t_2'. \tag{8}$$

What does this equation become in terms of $(\xi,y;t)$?

According to II, S must take the speed of light to be c. Therefore if t is the time, according to S's nearest clock, at which the light reaches x', this time must be equal to x/c, where x is S's measure of x''s position. So, using (6), we have

$$ct = x = \xi + vt$$

*Though, as the sequel will show, it really does not matter whether S or S' measures it.

or

$$t = \frac{\xi}{c - v}, \tag{9}$$

and (7) now gives

$$t_1' = l_3\xi + n_3\frac{\xi}{c - v}. \tag{10}$$

In a similar way, we find that the time t which corresponds to the return of the light to 0′ is given by

$$t = \frac{\xi}{c - v} + \frac{\xi}{c + v}. \tag{11}$$

And so, by (7) once more, the "ξ" in (7) now being zero because we are back at 0′, we have

$$t_2' = n_3\left(\frac{\xi}{c - v} + \frac{\xi}{c + v}\right). \tag{12}$$

On substituting from (10) and (12) into the synchronization equation (8) and simplifying, we find that

$$l_3 + n_3\frac{v}{c^2 - v^2} = 0. \tag{13}$$

If we perform a similar comparison of synchronizations along the y'-axis, it can be easily verified that we get

$$t_1' = m_3 y + n_3\frac{y}{\sqrt{c^2 - v^2}}, \quad t_2' = n_3\frac{2y}{\sqrt{c^2 - v^2}}, \tag{14}$$

which, in conjunction with (8), give simply

$$m_3 = 0. \tag{15}$$

From (15), (13), and (7), we now have

$$t' = n_3\left(t - \frac{v}{c^2 - v^2}\xi\right),$$

or, in terms of x rather than ξ,

$$t' = n_3\beta^2\left(t - \frac{v}{c^2}x\right), \tag{16}$$

where

$$\beta=\left(1-\frac{v^2}{c^2}\right)^{-1/2}, \tag{17}$$

a highly important quantity in relativity.

Instead of going back to the first equation of (7), we may take a short cut by considering the progress of the synchronizing light sent out from 0′. Because of II, its position along the x'-axis at any moment must be described by S and S' respectively as

$$x=ct, \tag{18}$$

$$x'=ct,' \tag{19}$$

where (x,t), (x',t') represent the same point and moment. We have already seen that (18) implies (9). On substituting in (19) from (16) and (9), using (6), and simplifying, we find

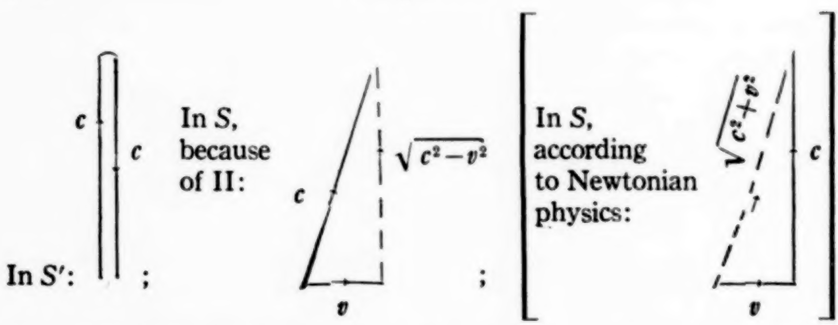
$$x'=n_3\beta^2(x-vt). \tag{20}$$

Next we describe the progress of light sent along the y'-axis. Here the analysis is a little different. According to S' we shall have

$$y'=ct'\ , \tag{21}$$

but, as the accompanying velocity diagrams show, according to S we must have

$$y=\sqrt{c^2-v^2}\ t\ . \tag{22}$$



From (7), (15), and the fact that ξ is here zero, we have

$$t'=n_3t. \tag{23}$$

Combining (21), (22), and (23), we at once obtain

$$y'=n_3\beta y. \tag{24}$$

Let us collect our three main results (16), (20), and (24), and see what they look like. If we write $\alpha(v)$ for $n_3\beta$, we may put them in the form

(25)
$$x' = \alpha(v)\beta(x - vt),$$
$$y' = \alpha(v)y,$$
$$t' = \alpha(v)\beta\left[t - \frac{v}{c^2}x\right].$$

All that remains is somehow to determine the function $\alpha(v)$, and then we shall have found the transformation we are seeking. Einstein solved this remaining problem in a very simple way. He introduced a third coordinate system S'', having axes parallel to those of S', and moving relative to it with velocity v in the negative x'-direction. To assume that S'' is the same as S would be highly improper since in one case v was measured by S, and in the other by S'. However, we may apply (25) in going from S' to S'' provided we change the sign of v—and, of course, put in appropriate primes. And if we do this, it turns out that the final transformation from S directly to S'' takes the simple form

(26)
$$x'' = \alpha(v)\alpha(-v)x; \; y'' = \alpha(v)\alpha(-v)y; \; t'' = \alpha(v)\alpha(-v)t.$$

Since the relationship between (x'',y'') and (x,y) does not contain t explicitly, the two systems S'' and S must be at rest relative to each other, and under the present circumstances must therefore be coincident. Therefore $\alpha(v)\alpha(-v)$ must have the value unity. But, as a moment's reflection will show, considerations of symmetry affecting the y-axes require that $\alpha(v)$ and $\alpha(-v)$ shall be equal. Hence we must have $\alpha(v) = \pm 1$. Moreover the (-1) may be ignored as contributing nothing essential that the $(+1)$ does not supply. Hence, finally, we arrive at the long awaited law of transformation between S and S':

(26)
$$x' = \beta(x - vt),$$
$$y' = y; \; z' = z,$$
$$t' = \beta\left[t - \frac{v}{c^2}x\right].$$

It is now a simple exercise in algebra to prove the important result that the inverse transformation is

(26′)
$$x = \beta(x' + vt'),$$
$$y = y'; \; z = z',$$

$$t=\beta\left[t'+\frac{v}{c^2}x'\right],$$

a result which one would expect on physical grounds.

The transformation (26), or (26′), was first discovered by Lorentz, and is therefore called the Lorentz transformation. However, Lorentz argued along entirely different and far less general lines, and his interpretation of the transformation was quite restricted. In fact, most of the implications we shall now deduce from the transformation cannot be legitimately derived in terms of Lorentz's ideas.

A. *Lengths.* Consider a rod lying at rest in S' along the x'–axis, with its ends at $x'=x_1'$, $x'=x_2'$. Its length is $(x_2'-x_1')$. At least that is what S' says. What does S think? If, as the rod rushes by, he notes that the ends, *as observed simultaneously at* $t=t_0$, are at $x=x_1$ and $x=x_2$, he must say the length is (x_2-x_1). But since S and S' never agree about simultaneity, they will disagree about the length of the rod. Specifically, we have, from (26), $x_2'=\beta(x_2-vt_0)$ and $x_1'=\beta(x_1-vt_0)$, whence by subtraction we find

$$(x_2'-x_1')=\beta(x_2-x_1). \tag{27}$$

Since, by (17), $\beta>1$, we see that the "rest length", $(x_2'-x_1')$, is greater than the length when the rod is moving past the observer. In other words, objects contract when in motion.* Note that c plays the part of a maximum relative velocity, because of the factor β in (27).

B. *Time Intervals.* Let a clock C, permanently at the fixed point $(x_0',0,0)$ in S', strike when it reads t_1', and again when it reads t_2'. Then, according to S', the time interval between the strokes is $(t_2'-t_1')$. S measures this time interval by first noting the reading of that particular one of his set of synchronous clocks which happens to be nearest to C when it first strikes, and then repeating the procedure (using, of course, *another* of his clocks) for the second stroke. If the two readings are t_1 and t_2, then S must maintain that the time interval is (t_2-t_1). By (26′),† we have

$$t_2=\beta\left[t_2'+\frac{v}{c^2}x_0'\right] \quad\text{and}\quad t_1=\beta\left[t_1'+\frac{v}{c^2}x_0'\right],$$

whence

$$(t_2-t_1)=\beta(t_2'-t_1'). \tag{28}$$

*The contraction here found is identical in amount with that proposed by Fitzgerald and Lorentz.

†(26′) is used here in preference to (26) because x_0' is constant and the algebra is easier when x_0' enters the transformation equation for the times explicitly.

That this shows that clocks in motion go more slowly than clocks at rest relative to an observer we shall leave for the reader to puzzle out.

To complete this picture of how lengths and time intervals behave we make one further remark. Because of I, it turns out that while S is insisting that S''s lengths have shrunk and his clocks slowed down, S' maintains just the same things about S!

C. *Addition of Velocities.* Let a point P move along the $x'-$axis with velocity u as measured by S'. We seek its velocity relative to S. Newtonian physics would say that it is $(u+v)$, but relativity gives a different answer.

If (x,t) and (x',t') describe the motion of P in S and S' respectively, then dx'/dt' will be u, and dx/dt will be the resultant velocity we seek. From (26'), $dx=\beta(dx'+v\,dt')$ and

$$dt=\beta\left[dt'+\frac{v}{c^2}dx'\right].$$

So, on dividing top and bottom of the ensuing fraction by dt', we have

$$\text{(29)}\quad \text{Resultant Velocity}=\frac{dx}{dt}=\frac{dx'+v\,dt'}{dt'+\frac{v}{c^2}dx'}=\frac{u+v}{1+\frac{uv}{c^2}}.$$

This relativistic law for the composition of velocities differs from the Newtonian law essentially because u and v are measured in different coordinate systems, this being the case of physical importance. Were u and v measured in the same system, the Newtonian law would prevail even in the theory of relativity. But to reduce v to its measure in S we had to use the Lorentz transformation, and all the peculiarities of length and time which go with it had full play. Try the effect of combining velocities u and c according to this formula. The result will give yet another illustration of the fact that c is a limiting velocity.

Naturally a theory which produces such startling results as those we have just found must cause a tremendous upheaval in physical science, and must provide many novel results capable of experimental investigation. The special theory of relativity has been most excellently confirmed by experiment, and has contributed to many branches of science. On the purely kinematical level, (27) and similar formulas explain many hitherto obscure points in optical theory, centering around the celebrated "drag coefficient" of Fresnel. Before we may proceed to the dynamical implications of the theory, we must mention, however briefly, the important aspect of the Lorentz transformation treated in the following section.

5. *The Four-Dimensional World.* It was the pure mathematician Minkowski who discovered that the Lorentz transformation has a singularly beautiful geometrical significance.

In the sense that four coordinates (x,y,z,t) are needed to specify the spatial and temporal location of an event relative to a reference system, we may look upon the world of physics as four-dimensional. This is not peculiar to relativity; it is also true of the Newtonian theory. Indeed, it is even true of both theories that the quantity

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 \tag{30}$$

is invariant under their respective types of transformations, as may be quickly verified. However, in Newtonian physics ds^2 is not a natural invariant but breaks up into two separately invariant parts, c^2dt^2 and the rest. In relativity (x,y,z) and t get well mixed together by a Lorentz transformation and ds^2 behaves as a single invariant. In fact, as Minkowski demonstrated, the transformation (26) is simply a rotation of coordinate axes about the xt-plane in the four dimensional space of (x,y,z,t), with ds the element of *interval* in this space just as $\sqrt{dx^2+dy^2+dz^2}$ is the element of *distance* in Euclidean space. Owing to the minuses in (30), the Minkowski space-time is hyperbolic, and the rotations involve hyperbolic instead of circular functions. For instance, (26) represents a rotation of coordinate axes in space-time through an "angle"

$$\tanh^{-1}\frac{v}{c},$$

while (29) is essentially the formula for $\tanh(\alpha+\beta)$ with

$$\tanh\alpha = \frac{u}{c} \text{ and } \tanh\beta = \frac{v}{c}.$$

The baffling contractions of length and dilations of time intervals also have a simple significance in the Minkowski world. They turn out to be nothing more than a foreshortening effect due to the difference in perspective between two sets of axes differently oriented in space time.

Minkowski's discovery put the whole of relativistic physics as then developed on a geometrical basis, and displayed the so-called theory of relativity as a theory of the absolute in a four-dimensional world. It was probably the most important contribution made to the theory of relativity by anyone except Einstein himself, and had a profound influence upon the subsequent course of the theory.

6. *Relativistic Dynamics.* The principle of relativity, I, especially in connection with the work of Minkowski, is of tremendous practical value, and will be a key principle in the present discussion of dynamics according to the special theory of relativity. The need for a relativistic dynamics is evident from the curious behavior of space, time, and velocity displayed in §4. It is made quite definite by the fact that the Newtonian equations (1) are not invariant under a Lorentz transformation, and thereby conflict with I. Our task is to construct alternative dynamical equations which shall accord with I, and shall yet differ from (1) as little as possible; for despite the many hard words we have uttered against the Newtonian equations, we must admit they are extraordinarily successful.

When we view this problem from the point of view of Minkowski, we find several suggestive ideas which are by no means obvious from any other viewpoint. For instance, if we are to write a vector equation in four-dimensional space, it must have four component equations, and not three. Again, since x, y, z, and t enter the Minkowski world on an almost equal footing, it is apparent that we may not initially use dx/dt, etc. in our equations since such expressions give to t a special distinction not accorded the other coordinates.

Let us consider a particle with space-time coordinates (x,y,z,t) moving with velocity v relative to the coordinate system. From (30) we find

$$\left(\frac{ds}{dt}\right)^2 = c^2 - \left\{\left(\frac{dx}{dt}\right)^2 - \left(\frac{dy}{dt}\right)^2 - \left(\frac{dz}{dt}\right)^2\right\} = c^2 - v^2,$$

whence

$$c\frac{dt}{ds} = \beta. \tag{31}$$

Thus for ordinary velocities, dt has practically the same value as

$$\frac{1}{c}\,ds,$$

and since ds is an invariant of space-time we may tentatively use

$$\frac{1}{c}\,ds$$

in place of dt in the Newtonian equations. The most natural four-dimensional modification of these equations along such lines is the set

$$(32)\qquad P_x=c^2\frac{d}{ds}\left[m\frac{dx}{ds}\right],\quad P_y=c^2\frac{d}{ds}\left[m\frac{dy}{ds}\right],$$

$$P_z=c^2\frac{d}{ds}\left[m\frac{dz}{ds}\right],\quad P_t=c^2\frac{d}{ds}\left[m\frac{dt}{ds}\right],$$

where m is a constant measuring the amount of matter in the body, and the entity (P_x,P_y,P_z,P_t), reminiscent of the Newtonian force vector (F_x,F_y,F_z), must transform like a four dimensional vector, such as (dx, dy, dz, dt), in order that Lorentz invariance shall be achieved. The four-vector (P_x,P_y,P_z,P_t) is not the force, and strictly speaking Newtonian terminology has no place in relativity. However scientists continue to use such terminology, and curious consequences of the theory theory of relativity arise from attempts to fit relativistic laws into classical moulds. In order to discuss the meaning of (32) we must make just such an attempt here.

When the particle is stationary relative to the coordinate system, ds has the value $c\,dt$, and the first three equations of (32) become the same as the Newtonian equations (1), with (P_x,P_y,P_z) corresponding to the Newtonian force. The fourth equation yields $P_t=m$, which has no significance for us at present.

Let us now consider in detail the appearance equations (32) take on when the particle is moving relative to the coordinate system with velocity v. To make comparison with Newtonian physics, we must first replace d/ds by d/dt by means of (31). This gives for (32)

$$(33)\qquad \frac{1}{\beta}P_x=\frac{d}{dt}\left\{m\beta\frac{dx}{dt}\right\},\quad \frac{1}{\beta}P_y=\frac{d}{dt}\left\{m\beta\frac{dy}{dt}\right\},$$

$$\frac{1}{\beta}P_z=\frac{d}{dt}\left\{m\beta\frac{dz}{dt}\right\},\quad \frac{1}{\beta}P_t=\frac{d}{dt}\{m\beta\}\,.$$

According to Newton, force is equal to the time rate of change of momentum. We must therefore interpret

$$\frac{1}{\beta}(P_x,P_y,P_z)$$

as the force and $m\beta\ (dx/dt,\ dy/dt,\ dz/dt)$ as the momentum. It is the latter which is of immediate interest; for if we wish to retain the New-

tonian nomenclature—as we do—and say that momentum is mass times velocity, then since the velocity is certainly $(dx/dt, dy/dt, dz/dt)$ we are obliged to interpret $m\beta$ as the mass. We write

$$M = m\beta, \tag{34}$$

and call M the relative mass. When v is different from zero the relative mass is larger than the "rest mass" m, and approaches an infinite value as v approaches the velocity of light c. Thus the faster a body moves the heavier it becomes, and by the time it attains a velocity c no finite force can accelerate it since it then has infinite mass. The kinematical principle that c is a limiting velocity in relativity here takes on a dynamical significance. Moreover formula (34) has been directly verified by experiment on moving electrons.

The fourth equation of (33) throws further light on the problem of mass. From (30) we have

$$c^2\left(\frac{dt}{ds}\right)^2 - \left(\frac{dx}{ds}\right)^2 - \left(\frac{dy}{ds}\right)^2 - \left(\frac{dz}{ds}\right)^2 = 1.$$

If we differentiate this with respect to s and then divide by $2dt/ds$, we obtain

$$c^2\frac{d^2t}{ds^2} = \frac{d^2x}{ds^2}\,\frac{dx}{dt} + \frac{d^2y}{ds^2}\,\frac{dy}{dt} + \frac{d^2z}{ds^2}\,\frac{dz}{dt}.$$

If we now multiply by mc^2, use (31), and replace

$$\frac{1}{\beta}(P_x,P_y,P_z) \text{ by } (F_x,F_y,F_z)$$

in accordance with the interpretation of the first three equations of (33), we find that

$$c^2\frac{P_t}{\beta} = F_x\frac{dx}{dt} + F_y\frac{dy}{dt} + F_z\frac{dz}{dt}.$$

The right-hand side has an immediate Newtonian interpretation. It is the scalar product of force and velocity, and therefore the rate at which work is being done on the particle by the force. Denoting the work by W, we may therefore write

$$c^2\frac{P_t}{\beta} = \frac{dW}{dt}.$$

This in itself is of little interest, but yields a result of the highest importance in conjunction with the last equation in (33). For the two together give

$$\frac{dW}{dt}=c^2\frac{d}{dt}(m\beta)=\frac{d}{dt}(Mc^2), \tag{35}$$

and this states that the rate work is done on the body, which, according to the law of conservation of energy, is the rate of change of its energy,* is equal to the rate of change of c^2 times its mass. Mass and energy must therefore be essentially the same sort of thing, and an increase ΔM in the mass of a particle will be equivalent to an increase $c^2\Delta M$ in i ts energy. This extraordinary doctrine receives an elegant mathematical confirmation if we consider a coordinate system in which the particle under discussion was initially at rest, and let a force acting on it ultimately give it a velocity v relative to this coordinate system. For then, by (34) and (17), we have

$$c^2\Delta M=c^2m\left\{\left[1-\frac{v^2}{c^2}\right]^{-\frac{1}{2}}-1\right\}=\tfrac{1}{2}mv^2+\cdots,$$

which, to a high degree of approximation is just the Newtonian expression for the increase in the kinetic energy of the particle.

When Einstein discovered the relationship between mass and energy, experimental confirmation was impossible. Nowadays, however, a quarter of a century later, conversions of matter into energy and energy into matter such as Einstein predicted are a routine observation in nuclear research, and Einstein's formula has been confimed up to the hilt.

Of the many other successful applications of the special theory of relativity, especially in the quantum theory, we may give no further indication here. There are other matters to be considered.

7. *The General Theory.* The success of the special theory of relativity convinced Einstein that other, more general things must also be true, and after ten arduous years he produced a general theory of relativity within which the special theory was a mere incident. Unfortunately not even the general theory can prevail against limitations of space—in the editorial sense—and we must be content here with but a brief indication of the scope of Einstein's monumental conception.

The Newtonian law of gravitation, not being invariant under a Lorentz transformation, clearly needed revision; and modified, Lorentz-

*Since the work done by the force is always equal to the energy gained by the body.

invariant laws were in fact not hard to find. Einstein alone among physicists saw that such simple modifications were superficial, and that a satisfactory theory of gravitation was to be obtained only through a profound generalization of the relativity concept.

The special theory of relativity had brought about a vital change in the status of the principle of relativity. What had hitherto been a more or less arbitrary, experimental principle of dynamics became, in Minkowski's four dimensional world, a geometrical commonplace. It said simply that one set of rectangular coordinates was as good as another for describing four dimensional geometry. Now why should one stop at rectangular coordinates? Surely a geometrical configuration must be independent of whatever coordinates are used to describe it, whether rectangular or curvilinear. *All* coordinate systems must be equivalent for describing the geometry of space-time.

This simple geometrical commonplace takes on a tremendous significance when translated into the language of physics. It requires that

III. *All possible coordinate systems (that is, coordinate scaffoldings together with their sets of clocks) must be equivalent for the expression of physical laws.*

And this generalized principle of relativity immediately stubs its toe against one of the most elementary facts of dynamical experience.

For, according to III, an accelerated system must be equivalent to one that is unaccelerated. But, as we saw in §1, experience tells us otherwise. Free bodies do not move in straight lines with uniform velocity relative to accelerated systems and we can thus clearly distinguish between accelerated and unaccelerated ones. What hope is there, then, for the survival of III?

There is still hope. Let us consider what effect the acceleration of a system produces. It causes a relative acceleration of all bodies independent of their masses. Does this not stir up memories of Galileo and Pisa? By a most extraordinary piece of good fortune, it happens that gravitation produces precisely the same effect. We have no need now of giving up III. If someone asserts that his coordinate system is accelerated, we can blandly inform him he is entirely mistaken; that his system is unaccelerated, and that the accelerational effects on which he bases his claims are in fact the manifestations of a gravitational field. No experiments he may perform can decide the issue for him. We have identified his *fictitious forces* as gravitational forces, which produce identical effects. Our position is impregnable, and we reach the momentous conclusion that if a theory is to be constructed in

accordance with III, it must embrace gravitation. There is a significant and encouraging fact to be considered in this connection. According to all previous concepts it is indeed an extraordinary piece of good fortune that gravitational and accelerational effects should be so conveniently identical.* In a theory of gravitation based on III, their identity must cease to be a mystery for it will become a simple matter of definition.

The precise place where gravitation must enter the mathematics of such a theory is clearly indicated, for it must occur wherever acceleration makes an appearance. Therefore let us consider the formula for the element of interval ds in space-time. A Lorentz transformation leaves the form (30) invariant. A transformation representing an acceleration, however, spoils the simplicity of the form. From a consideration of relatively rotating coordinate systems, Einstein was able to demonstrate that the (x,y,z,t) measured by rods and clocks can no longer be regarded as necessarily rectangular, rectilinear coordinates having a direct metrical significance, but must be looked upon as general, Gaussian coordinates of space-time. From this he pointed out that ds^2 must be a general quadratic form

$$(35) \qquad ds^2 = \sum_{a,b=1}^{4} g_{ab}dx^a dx^b, \quad (dx^1,dx^2,dx^3,dx^4) \sim (dx,\ dy,\ dz,\ dt),$$

the coefficients g_{ab} being in general functions of position and time. In (30), g_{11}, g_{22}, g_{33}, each had the value (-1), and g_{44} had the value c^2, while all others, such as g_{12}, were zero. Acceleration of the coordinate system has the effect of altering these specially simple values of the g's, and inasmuch as acceleration shows itself in these g's, it follows that they must also be the seat of gravitational phenomena. The g's in the general theory of relativity characterize simultaneously the type of coordinate system used and the gravitational field present. In order that ds shall be invariant under all transformations—and without some such invariant measurement could not be defined—the g's must be the components of a second rank covariant tensor, and they form in fact the familiar metric tensor of Riemannian geometry.

There still remains the problem of setting up field equations of gravitation to replace the Newtonian law, and here we find one of the most beautiful aspects of the general theory of relativity. For, if one requires simply that these field equations shall be tensor equations containing only the g's and their first and second derivatives, and that certain divergence relations between them shall be fulfilled which correspond to the laws of conservation of momentum and energy and at

*In Newtonian parlance, this is because the gravitational and inertial masses of a body are always proportional. No explanation for this is possible in Newtonian theory, yet it is a very well attested experimental fact.

the same time insure that non-trivial solutions of the equations shall exist, then the field equations are essentially ***uniquely determined.*** This is a far cry from the arbitrariness of the Newtonian law of gravitation.

The field equations to which one is thus inevitably led turn out to involve the Riemann-Christoffel curvature tensor, and imply that gravitation is but the manifestation of a curvature of space-time. And with gravitation taking on this geometrical significance we find Minkowski's geometrical ideas carried to a conclusion Minkowski was unable to foresee. The great Riemann himself, as far back as 1850, had sought such a theory of gravitation and other phenomena, but his attempt, as we now realize, was foredoomed to failure for he did not know the world is truly four dimensional.

The astronomical predictions of the general theory of relativity and their successful confirmation are well known. But the greatness of the theory rests on a surer foundation than just this, for it also removes many of the major philosophical difficulties hitherto associated with physical theory. As a field theory it avoids the objectionable concept of action at a distance. It deposes space and time from their lofty thrones and makes them subject to experiment instead of axiom, and it shows that experiment decides definitely against the spurious claims of Newtonian time and Euclidean space. Its laws are invariant under all transformations of coordinates, among which are those of the type $x' = x$, $y' = y$, $z' = z$, $t' = f(t)$. It therefore gives no philosophical preference to a specially "accurate" chronometer over and above any quite variable watch, and so avoids the philosophical problem of trying to define an absolute standard of time which shall flow uniformly. It likewise avoids the similar problems connected with defining standards of length which shall be forever invariable. The standards of length and time we set up are merely conveniences in the spreading of a web of coordinate lines over space-time with which to ensnare the elusive true invariants of the world; not standards of length and time, but the motion of light, the curvature of the world, the paths of planets, and other things not the invention of man. And this is indeed the whole of the general principle of relativity, that physical phenomena can have no concern for our whims and preferences for one system or another but must transcend all such petty considerations. Here is to be found the ultimate reason for general invariance, and here, in the very naturalness and simplicity of its basic assumption, resides the real triumph of the theory of relativity. The enormous range of the theory, its striking success, and its mathematical and philosophical elegance and economy combine to place it among the noblest creations of the human mind, and proclaim its creator one of the greatest scientists in the history of the world.

Applications of Complex Numbers to Geometry of Circles

By Allen A. Shaw
University of Arizona

A few years ago the present writer wrote two papers on applications of complex numbers to geometry of the straight line and polygon.* He now wishes to extend the theory to the geometry of circles.

In this article we shall make use of some of the following simple formulas:

The distance from the point $P_1(z_1)$ to the point $P_2(z_2)$ is given by

(1) $$P_1P_2 = |z_2 - z_1|.$$

The point $P(z)$ which divides in a given ratio r the stroke from z_1 to z_2 is given by

(2) $$z = \frac{z_1 + rz_2}{1+r},$$

where r is real.

When $r=1$ we have, as a special case, *the mid-point formula*:

(2′) $$z = \tfrac{1}{2}(z_1 + z_2).$$

When $r=2$, we have *the trisection-point formula*:

(2″) $$z = \tfrac{1}{3}(z_1 + 2z_2)$$

(3) *kz and z, $k(z_1 \pm z_2)$ and $(z_1 \pm z_2)$, kz_1z_2 and z_1z_2,*

$$k\,\frac{z_1}{z_2} \text{ and } \frac{z_1}{z_2},$$

all represent parallel vectors in pairs, where k is real. In each pair the lengths are in the ratio $k:1$.

(4) (i) $z\bar{z} = r^2 = x^2 + y^2 = |z|^2$;

(ii) *if z lies on the axis of real, $z = \bar{z}$.*

*See A. A. Shaw, *Geometric Applications of Complex Numbers*: School Science and Mathematics, Vol. 31, (1931), pp. 754-761, and *Applications of Complex Numbers to Geometry*: The Mathematics Teacher, April, 1932, pp. 215-226.

LEMMA: The necessary and sufficient condition that four points be concyclic is that their cross ratio be real.

(a) *The condition is necessary.*

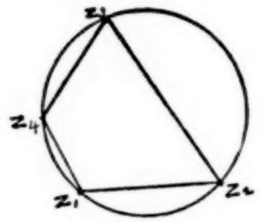


Fig. 1.

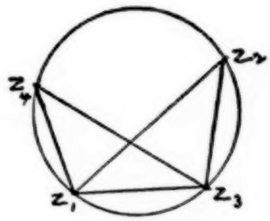


Fig. 2.

Proof: Given that z_1, z_2, z_3, z_4 are concyclic.

To prove that the cross ratio $\left(\dfrac{z_1-z_2}{z_3-z_2}\cdot\dfrac{z_3-z_4}{z_1-z_4}\right)$ is real.

Since z_1, z_2, z_3, z_4 are concyclic,

Then $\angle z_3z_2z_1+\angle z_1z_4z_3=0^\circ$ or 180° (Fig. 1 and 2),

or $$am\left(\frac{z_1-z_2}{z_3-z_2}\right)+am\left(\frac{z_3-z_4}{z_1-z_4}\right)=0 \text{ or } \pi,$$

i. e. $$am\left(\frac{z_1-z_2}{z_3-z_2}\cdot\frac{z_3-z_4}{z_1-z_4}\right)=0 \text{ or } \pi.$$

Therefore, in either case, the cross ratio

$$\left(\frac{z_1-z_2}{z_3-z_2}\cdot\frac{z_3-z_4}{z_1-z_4}\right)$$

is real positive in the first case and negative in the second.

(b) *The condition is sufficient.*

Proof: Given that the cross ratio $\left(\dfrac{z_1-z_2}{z_3-z_2}\cdot\dfrac{z_3-z_4}{z_1-z_4}\right)$ is real,

To prove that the four points z_1, z_2, z_3, z_4 are concyclic.

Proof: Since $\left(\dfrac{z_1-z_2}{z_3-z_2}\cdot\dfrac{z_3-z_4}{z_1-z_4}\right)$ is real (positive or negative),

Then $$am\left(\frac{z_1-z_2}{z_3-z_2}\cdot\frac{z_3-z_4}{z_1-z_4}\right)=0 \text{ or } \pi,$$

or $$am\left(\frac{z_1-z_2}{z_3-z_2}\right)+am\left(\frac{z_3-z_4}{z_1-z_4}\right)=0 \text{ or } \pi,$$

i. e. $$\angle z_3z_2z_1+\angle z_1z_4z_3=0^\circ \text{ or } 180^\circ \quad \text{(Fig. 1 and 2)}.$$

Therefore, in either case, the points are concyclic.

Theorem 1: *Inscribed angles upon the same chord and on the same side of it are congruent.*

Given that A, B, C, D are concyclic and C and D on the same side of the chord AB,

To prove that $\angle C=\angle D$.

Proof: Since A, B, C, D are concyclic,

Then the cross ratio $\left(\dfrac{z_2-z_3}{z_1-z_3}\div\dfrac{z_2-z_4}{z_1-z_4}\right)$

is real by Lemma (a), and positive,*

Hence $$am\left(\frac{z_2-z_3}{z_1-z_3}\right)-am\left(\frac{z_2-z_4}{z_1-z_4}\right)=0,$$

or $$\angle z_1z_3z_2=\angle z_1z_4z_2. \quad \text{(Fig. 3)}$$

Therefore $$\angle C=\angle D.$$

Fig. 3.

Theorem 2: *Conversely if the line joining two points subtend equal angles at two other points on the same side of it, then the four points are concyclic.*

*The cross ratio (z_2z_1,z_3z_4) is positive since the displacement $\overrightarrow{z_2z_1}$ does not contain the points z_3 or z_4. For the sign of the cross ratio see J. L. S. Hatton: *Principles of Projective Geometry*, p. 17. Cambridge Univ. Press; or W. C. Graustein: *Higher Geometry*, p. 75, The Macmillan Company.

Given that $\angle ACB = \angle ADB,$

To prove that A, B, C, D are concyclic.

Proof: If $\angle ACB = \angle ADB,$

Then $$am\left(\frac{z_2-z_3}{z_1-z_3}\right) - am\left(\frac{z_2-z_4}{z_1-z_4}\right) = 0,$$

or $$am\left(\frac{z_2-z_3}{z_1-z_3} : \frac{z_2-z_4}{z_1-z_4}\right) = 0,$$

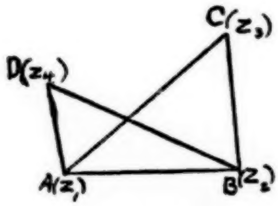


Fig. 4.

Therefore the cross ratio $\left(\frac{z_2-z_3}{z_1-z_3} : \frac{z_2-z_4}{z_1-z_4}\right)$ is real and positive.

Hence the points z_1, z_2, z_3, z_4 are concyclic.*

Theorem 3: *In a cyclic quadrilateral opposite angles are supplementary.*

Given that $ABCD$ is a cyclic quadrilateral,

To prove that $A+C=B+D=180°$.

Proof: Since A, B, C, D are concyclic,

Then the cross ratio $\left(\frac{z_1-z_2}{z_3-z_2} \cdot \frac{z_3-z_4}{z_1-z_4}\right)$

is real by Lemma (a), and is negative,†

Hence $$am\left(\frac{z_1-z_2}{z_3-z_2} \cdot \frac{z_3-z_4}{z_1-z_4}\right) = \pi,$$

or $$am\left(\frac{z_1-z_2}{z_3-z_2}\right) + am\left(\frac{z_3-z_4}{z_1-z_4}\right) = \pi,$$

*This theorem can be generalized. See Heath's Euclid, Vol. II. pp. 50-51.

†Since the displacement z_1z_2 contains z_3 but not z_4. See Hatton, *loc. cit.*, or Graustein, *loc. cit.*

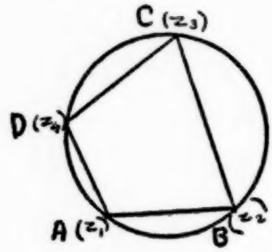


Fig. 5.

i. e. $$\angle z_3z_2z_1 + \angle z_1z_4z_3 = 180°. \qquad \text{(Fig. 5).}$$

Similarly it can be shown that

$$am\left(\frac{z_4-z_1}{z_2-z_1}\right) + am\left(\frac{z_2-z_3}{z_4-z_3}\right) = \pi,$$

and the theorem follows.

Theorem 4: *Conversely, if two opposite angles of a quadrilateral are supplementary, then it is cyclic.*

Given that $$A+C=B+D=180°,$$

To prove that A, B, C, D are concyclic.

Proof: Since $$B+D=180°,$$

Then $$am\left(\frac{z_1-z_2}{z_3-z_2}\right) + am\left(\frac{z_3-z_4}{z_1-z_4}\right) = \pi.$$

Hence the cross ratio $\left(\dfrac{z_1-z_2}{z_3-z_2} \cdot \dfrac{z_3-z_4}{z_1-z_4}\right)$ is real and negative.

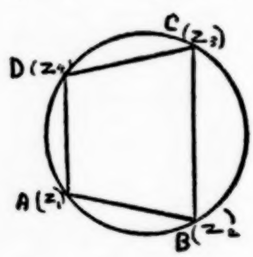


Fig. 6.

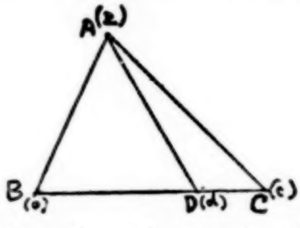
Therefore the points A,B,C,D are concyclic by Lemma (b).

Theorem 5: *Stewart's Theorem: The square of the distance of a point on the base of a triangle from the opposite vertex multiplied by the base is equal to the sum of the squares of the other two sides, each multiplied by the non-adjacent segment of the base, minus the product of these two segments multiplied by the base.*

i. e. Prove that

(1) $$AD^2 \cdot BC = BD \cdot AC^2 + DC \cdot AB^2 - BC \cdot BD \cdot DC$$

Proof: Writing $(z-d)(\bar{z}-d)$ for DA^2, $(z-c)(\bar{z}-c)$ for CA^2, $z\,\bar{z}$ for BA^2 etc. by formula 2, and taking B as origin, we have from (1)



(2) $$(z-d)(\bar{z}-d)c = d(z-c)(z-c) + (c-d)z\,\bar{z} - cd(c-d)$$

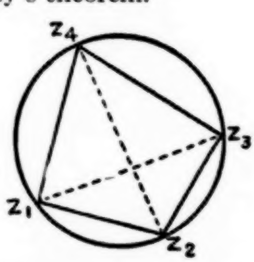
which is identically true and the theorem is proved.

Theorem 6(a): *Ptolemy's Theorem: In a cyclic quadrilateral the product of the diagonals is equal to the sum of the products of the opposite sides.*

Proof: Consider the identity

(1) $$(z_1-z_2)(z_3-z_4)+(z_1-z_3)(z_4-z_2)+(z_1-z_4)(z_2-z_3)=0$$

where the points z_1, z_2, z_3, z_4 are *consecutive* around the circle (in either direction) and are supposed *distinct*, so that no factor $z_i - z_j$ is zero. The join of *alternate* vertices, z_1-z_3 and z_4-z_2 are the *diagonals* definitely in the sense of Ptolemy's theorem.*



*The reader of projective geometry will recall that four points A,B,C,D are called a *quadrangle* and their six junctions, AB,CD; AC,BD; AD,BC are the three pairs of opposite *sides*. No pair of opposite sides is distinguished from others: there are no diagonals. But if the four points are concyclic the two lines joining the pairs of *alternate* vertices are called *diagonals* in the sense of the above theorem.

From (1) we have

$$(1) \qquad -1=\frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_3)(z_4-z_2)}+\frac{(z_1-z_4)(z_2-z_3)}{(z_1-z_3)(z_4-z_2)}$$

$$(2) \quad \text{or } 1=\left(\frac{z_1-z_2}{z_4-z_2}\div\frac{z_1-z_3}{z_4-z_3}\right)+\left(\frac{z_1-z_4}{z_2-z_4}\div\frac{z_1-z_3}{z_2-z_3}\right).$$

Now $$am\,\frac{z_1-z_2}{z_4-z_2}=\angle z_4z_2z_1,\quad am\,\frac{z_1-z_3}{z_4-z_3}=\angle z_4z_3z_1.$$

Since these angles are described in the same arc and in the *same* sense, they are equal.

Hence $$\frac{z_1-z_2}{z_4-z_2}:\frac{z_1-z_3}{z_4-z_3} \text{ is a \textit{positive real} number,}$$

$$\therefore \qquad \frac{z_1-z_2}{z_1-z_3}\cdot\frac{z_3-z_4}{z_4-z_2} \text{ is a \textit{negative real} number.}$$

Hence $(z_1-z_2(z_3-z_4)$ and $(z_1-z_3)(z_4-z_2)$ are on *opposite* sides of the origin, or their amplitudes differ by π.

Similarly $$\frac{z_1-z_4}{z_1-z_3}\cdot\frac{z_2-z_3}{z_4-z_2} \text{ is a \textit{negative real} number.}$$

Hence $(z_1-z_4)(z_2-z_3)$ and $(z_1-z_3)(z_4-z_2)$ are on opposite sides of the origin, or their amplitudes differ by π.
Consequently

$$(z_1-z_2)(z_3-z_4),\ (z_1-z_4)(z_2-z_3),\ (z_1-z_3)(z_4-z_2)$$

are all collinear on a line passing through the origin: the first two are on one side of the origin and the third on the other. Then from (1) we have

$$|-1|=\left|\frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_3)(z_4-z_2)}+\frac{(z_1-z_4)(z_2-z_3)}{(z_1-z_3)(z_4-z_2)}\right|,$$

$$=\left|\frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_3)(z_4-z_2)}\right|+\left|\frac{(z_1-z_4)(z_2-z_3)}{(z_1-z_3)(z_4-z_2)}\right|$$

since both terms of the right member of (1) have the *same* amplitude, π. or, $(z_1-z_3)\cdot(z_4-z_2)=(z_1-z_2)\cdot(z_3-z_4)+(z_1-z_4)\cdot(z_2-z_3)$ which proves the theorem:

Theorem 6(b). *Conversely, if the product of the diagonals of a quadrilateral is equal to the sum of the products of the opposite sides, the quadrilateral is cyclic.*

Given four points z_1, z_2, z_3, z_4 such that

$$(z_1-z_3)\cdot(z_4-z_2)=(z_1-z_2)\cdot(z_3-z_4)+(z_1-z_4)\cdot(z_2-z_3).$$

Prove that z_1, z_2, z_3, z_4 are concyclic.

From identity (1) in Theorem 6(a) we have

$$\frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_3)(z_4-z_2)}+1+\frac{(z_1-z_4)(z_2-z_3)}{(z_1-z_3)(z_4-z_2)}=0. \tag{2}$$

Hence the first fraction and the second fraction are two complex numbers whose sum is -1.

$$\text{But} \quad \text{mod.}\frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_3)(z_4-z_2)}+\text{mod.}\frac{(z_1-z_4)(z_2-z_3)}{(z_1-z_3)(z_4-z_2)}=1,$$

because the hypothesis says that

$$\text{mod}(z_1-z_3)(z_4-z_2)=\text{mod}(z_1-z_2)(z_3-z_4)+\text{mod}(z_1-z_4)(z_2-z_3).$$

Hence the modulus of the sum of the fractions (which is -1) is equal to the sum of their moduli. Hence the points representing the fractions are collinear with the origin and on the same side of it.

Hence, since the sum of the fractions is -1, they are both real negative numbers.

But the identity (1) may be written

$$-\left[\frac{z_1-z_2}{z_4-z_2}\div\frac{z_1-z_3}{z_4-z_3}\right]+1-\left[\frac{z_1-z_4}{z_2-z_4}\div\frac{z_1-z_3}{z_2-z_3}\right]=0$$

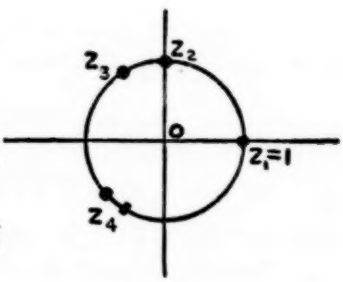
that is the quantities in square brackets, which are cross ratios, are real positive numbers.

Therefore the points are concyclic.

In fact, it follows that

$$\angle z_1z_2z_4=\angle z_1z_3z_4, \text{ and } \angle z_1z_4z_2=\angle z_1z_3z_2.$$

Hence, the points are concyclic in the order z_1, z_2, z_3, z_4.



Consider the following illustration of four points on the unit circle.

$$z_1 = 1$$
$$z_2 = i$$
$$z_3 = w = -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{3}i$$
$$z_4 = -\tfrac{1}{2}\sqrt{2} - \tfrac{1}{2}\sqrt{2}i$$
$$z_1 - z_2 = 1 - i$$
$$z_3 - z_4 = \tfrac{1}{2}\,[\sqrt{2} - 1 + (\sqrt{2} + \sqrt{3})i]$$

Hence $(z_1 - z_2)(z_3 - z_4) = \tfrac{1}{2}\{2\sqrt{2} + \sqrt{3} - 1 + (1 + \sqrt{3})i\}$,

$$\left.\begin{aligned} z_1 - z_3 &= \tfrac{1}{2} + 1 - \tfrac{1}{2}\sqrt{3}i \\ z_4 - z_2 &= -\tfrac{1}{2}\sqrt{2} - (\tfrac{1}{2}\sqrt{2} + 1)i \end{aligned}\right\}$$

Hence $(z_1 - z_3)(z_4 - z_2) = \tfrac{3}{4}\sqrt{2} - \tfrac{1}{4}\sqrt{6} - \tfrac{1}{2}\sqrt{3} + [\tfrac{1}{4}\sqrt{6} - \tfrac{3}{4}\sqrt{2} - \tfrac{3}{4}]i$

$$\left.\begin{aligned} (z_1 - z_4) &= \tfrac{1}{2}\sqrt{2} + 1 + \tfrac{1}{2}\sqrt{2}i \\ (z_2 - z_3) &= \tfrac{1}{2} + (1 - \tfrac{1}{2}\sqrt{3})i \end{aligned}\right\},$$

and $(z_1 - z_4)(z_2 - z_3) = (\tfrac{1}{2} - \tfrac{1}{4}\sqrt{2} + \tfrac{1}{4}\sqrt{6} + [\tfrac{3}{4}\sqrt{2} + 1 - \tfrac{1}{4}\sqrt{6} - \tfrac{1}{2}\sqrt{3}]i$. Hence mod $(z_1 - z_3)(z_4 - z_2)$ is $\sqrt{6 + 3\sqrt{2}}$; the moduli of the other two products

are $$\sqrt{4 - \sqrt{2} + \sqrt{6}} \text{ and } \sqrt{4 + 2\sqrt{2} - 2\sqrt{3} - \sqrt{6}};$$

letting $$\sqrt{6 + 3\sqrt{2}} = \sqrt{4 - \sqrt{2} + \sqrt{6}} + \sqrt{4 + 2\sqrt{2} - 2\sqrt{3} - \sqrt{6}},$$

transposing the first radical on the right and squaring

$$6 + 3\sqrt{2} + 4 - \sqrt{2} + \sqrt{6} - 2\sqrt{(6 + 3\sqrt{2})(4 - \sqrt{2} + \sqrt{6})} = 4 + 2\sqrt{2} - 2\sqrt{3} - \sqrt{6},$$

or $$\sqrt{(6 + 3\sqrt{2})(4 - \sqrt{2} + \sqrt{6})} = 3 + \sqrt{3} + \sqrt{6},$$

squaring

$$24 - 6\sqrt{2} + 6\sqrt{6} + 12\sqrt{2} - 6 + 6\sqrt{3} = 9 + 3 + 6 + 6\sqrt{3} + 6\sqrt{6} + 6\sqrt{2},$$

or $$18 + 6\sqrt{2} + 6\sqrt{6} + 6\sqrt{3} = 18 + 6\sqrt{2} + 6\sqrt{6} + 6\sqrt{3}.$$

Theorem 8: *In a non-cyclic quadrilateral the sum of the products of the opposite sides is greater than the product of the diagonals.*

Proof: This is an immediate consequence of Theorems 6a and 6b, for consider the identity as before

$$(1) \qquad (z_1 - z_2)(z_3 - z_4) + (z_1 - z_3)(z_4 - z_2) + (z_1 - z_4)(z_2 - z_3) = 0.$$

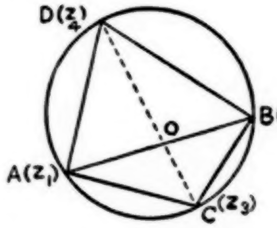
Since the points z_1, z_2, z_3, z_4 are not concyclic, the *resulting* cross ratios (see Theorems 6a and 6b above) will be complex. Consequently the

amplitudes of the three products of differences in (1) will be different, and we have

$$|(z_1-z_3)(z_4-z_2)| = |(z_1-z_2)(z_3-z_4)+(z_1-z_4)(z_2-z_3)|.$$

$$\therefore |z_1-z_3|\cdot|z_4-z_2| \angle |z_1-z_2|\cdot|z_3-z_4|+|z_1-z_4|\cdot|z_2-z_3|$$

which proves the Theorem.



Theorem 9: *If $AC:CB=-AD:DB$, and if A, B, C, D are the points z_1, z_2, z_3, z_4, show that* (i) *A, B, C, D are concyclic, and* (ii) $(z_1+z_2)(z_3+z_4)=2(z_1z_2+z_3z_4)$, *also prove* (iii) *$\Delta AOC \sim \Delta DOA$, where O is the midpoint of AB.*

Proof: Since $AC:CB=-AD:DB$,

Then
$$\frac{\overrightarrow{AC}}{\overrightarrow{BC}} \div \frac{\overrightarrow{AD}}{\overrightarrow{BD}} = -1$$

or

(1)
$$\frac{(z_3-z_1)\cdot(z_4-z_2)}{(z_3-z_2)\cdot(z_4-z_1)} = -1.$$

Hence
$$am\left(\frac{z_3-z_1}{z_3-z_2}\right) + am\left(\frac{z_4-z_2}{z_4-z_1}\right) = \pi,$$

or
$$\angle z_2z_3z_1 + \angle z_1z_4z_2 = 180°,$$

$\therefore$ the points A, B, C, D are concyclic.

(ii) From (1) we have

$$(z_3-z_1)(z_4-z_2) = -(z_3-z_2)(z_4-z_1)$$

which gives
$$2z_1z_2+2z_3z_4 = z_3z_2+z_1z_3+z_1z_4+z_2z_4$$
$$= z_3(z_2+z_1)+z_4(z_1+z_2),$$

or

(2)
$$2(z_1z_2+z_3z_4) = (z_1+z_2)(z_3+z_4)$$

which proves part (ii).

(iii) Taking the origin at the midpoint of AB, we have

$$z_1+z_2=0,$$

then from (2) we have

$$z_3z_4=-z_1z_2=z_1^2=z_2^2.$$

$\therefore$ (α) $\qquad \dfrac{z_4}{z_1}=\dfrac{z_1}{z_3} \qquad \therefore \quad \dfrac{\overrightarrow{Oz_4}}{\overrightarrow{Oz_1}}=\dfrac{\overrightarrow{Oz_1}}{\overrightarrow{Oz_3}}$

$$\therefore \Delta z_1Oz_4 \sim \Delta z_3Oz_1$$

and (β) $\qquad \dfrac{z_4}{z_2}=\dfrac{z_2}{z_3} \qquad \therefore \quad \dfrac{\overrightarrow{Oz_4}}{\overrightarrow{Oz_2}}=\dfrac{\overrightarrow{Oz_2}}{\overrightarrow{Oz_3}},$

$$\therefore \Delta z_2Oz_4 \sim \Delta z_3Oz_2. \qquad \text{Q.E.D.}$$

Humanism and History of Mathematics

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On the Discovery of the Logarithmic Series and Its Development in England up to Cotes

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To the expert of today the logarithmic series appears to be a very non-essential detail. In its time it was a very notable discovery as regards itself alone, as well as in the framework of the general theory of series. It was discovered *circa* 1667 by Newton and independently by Mercator. Huygens and Gregory were close to the same discovery but they were both anticipated by the other two. Newton was then 24 years old, Mercator 47. For Newton the logarithmic series was a beginning, for Mercator the climax.

I. *Nicolas Mercator (1620-1687)*. Mercator's life work is almost forgotten today, certainly unjustly. Mercator was a distinguished mathematician, physicist and astronomer. Shortly after his arrival in London the much-traveled man was received into the Royal Society. Products of that period are his new astronomical theory,[1] the edition of Euclid,[2] the navigation problems[3] and the calculation of logarithms.[4] We shall be concerned here with the latter.

The *Logarithmo-technia* is divided into three very unequal parts. The first two sections, which had already been published separately in 1667,[5] are devoted entirely to the calculation of a system of common logarithms. In the presentation of logarithms Mercator proceeds very intuitively and clearly according to the then generally customary usage. He divides the number domain between 1 and 10 by insertion of geometric means (he calls them *ratiunculæ*) into 10 million parts.[6] Thus the logarithm of a number between 1 and 10 is determined from the number of *ratiunculæ* between 1 and this number. Mercator now

develops his process for the calculation of the common logarithm of two bases (as such he chooses, stated in modern form, 1.005 and 0.995). It is very carefully thought out and in contrast to the previously adopted methods of calculation has the great advantage that the calculation is purely rational. By continued squaring Mercator gets two successive second powers of the base, between which is the number 10. Now the smaller of these powers is multiplied by the descending series of the previous squares until the new power exceeds 10. Then the performance is repeated until one has the two successive integral powers of the base, between which lies the number 10. Now several further decimal places of the power of the base are determined, which becomes approximately equal to 10, by means of the *regula falsi*. Hence there follows by division the number of the *ratiunculæ* which are referred to the base, *i. e.*, its logarithm.[7]

Now Mercator introduces the absolute value of the logarithm from the ratio of two positive magnitudes as its "proportional measure"[8] and teaches calculation with these proportional measures. Then he forms logarithms from the successive members of an arithmetical series and shows that the terms of their difference series become smaller and smaller. He builds up the logarithms themselves from the first terms of the difference series. The logarithms of powers and roots are approximated by means of the logarithms of rational approximation values. There is attached an excellent process for the gradual refinement of the calculation. As a supplement several formulas are given which serve convenience in calculation. Then follow more exact directions for the practical calculation of a complete table of logarithms.

In the third section the ordinate

$$y=\frac{1}{1+x}=1-x+x^2-x^3\cdots$$

of the equilateral hyperbola is transformed, by dividing out, into a power series. The surface of the hyperbola segment is built up entirely in the sense of Cavalieri's method of infinitesimals from the totality of all parallel coordinates "contained" in it. How one is to take and combine the sums over the single powers of x is only briefly alluded to.[9] Thus Mercator gets the hyperbolic segment in that form which we would today write thus:

$$\int_0^x \frac{dt}{1+t}=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}\cdots$$

However he does not express his result by a formula, but entirely in words. By means of an extremely bold conclusion he finds that this series can also be expressed by the logarithm of $(1+x)$. The paper ends with the calculation of the body which "consists" of infinitely many hyperbolic segments. We would thus write the result expressed again only in words:

$$\int_0^x \log(1+t)\cdot dt = \frac{x^2}{1\cdot 2} - \frac{x^3}{2\cdot 3} + \frac{x^4}{3\cdot 4} - \frac{x^5}{4\cdot 5} \cdots$$

This third section of the *Logarithmotechnia* was at once announced by John Wallis through a review in the *Phil. Trans.*, 10, in which the formal notation is very much improved. Wallis calls attention to the fact that Mercator's development is admissible only for $x<1$ and adds the development of

$$\int_0^x \frac{dt}{1-t} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots \qquad (0<x<1)$$

In a note by Mercator himself[11] the logarithms determined from the hyperbolic segments are expressly designated as "natural logarithms". Here the values of log 2, log 3, log 10 and log 11 are determined from the correctly combined series for

$$\log(1+x) \text{ and } \log\frac{1}{1-x}, \quad \text{with } x=0.1 \text{ and } 0.2.$$

Next Mercator by means of multiplication with

$$0.43429 = \frac{1}{\log 10}$$

transforms from the natural to the common logarithms and *vice versa*.

II. *James Gregory (1638-1675)*. A few months later Gregory came before the public with his *Exercitationes geometricæ* (London, 1668), whose interesting preliminary history I must briefly go into. On account of his *Vera circuli et hyperbolæ quadratura* (Padua, 1667), in which he sought to prove the algebraic impossibility of squaring a circle Gregory had involved himself in a heated quarrel with Huygens[12]. The very adverse, indeed unjust criticism of Huygens in the *Journal des Sçavans* (July, 1668) was followed by a rather irritable reply by Gregory in the *Phil. Trans.* (July, 1668), then a reply, unbearable for

Gregory, by Huygens in the *Journal des Sçavans* (Nov. 1668) and a brisk ccrrespondence between Huygens and the most influential members of the Royal Society, who would gladly have smoothed out the affair and thus hindered a further explanation by Gregory. The latter fell into frightful excitement. He believed he was persecuted, disliked and slighted on all sides. The *Exercitationes geometricæ*, on which he was just then working and in which he took a stand on numerous questions of the day[13], were misused in the foreword for a huge counter-attack against Huygens. Later the quarrel was put aside by the Royal Society—one might almost say, by compulsion—and Gregory's book was rather hushed up on account of the malicious introduction. This fate was undeserved; for the mathematical content of the *Exercitationes* is important.

For us only a small part of the little work is important.[14] There Gregory, who depends on the geometric integration method of Gregoire de Saint Vincent[15] and most probably was familiar with Fermat's essay *De æquationum localium transmutatione, & emendatione* · · ·[16], gives a completely impeccable proof for the representation of the hyperbolic segment by means of the power series. Moreoever he combines two adjacent segments and forms that series which we represent today by

$$\tfrac{1}{2}\log\frac{1+x}{1-x},$$

but everything is still expressed very ponderously. The relation to logarithms is only cursorily touched on and dismissed rather superficially. Meanwhile, we have a letter of Gregory of a later date, which gives us a better insight into his accomplishments.[17] Indeed there a subdominant (Minorante) is set up for the logarithmic series. It arises by setting from any member of the series for

$$\tfrac{1}{2}\log\frac{1+x}{1-x}$$

on, instead of the correct terms of the series those of a geometric series.[18]

III. *Isaac Newton (1642-1727)*. The logarithmic series plays a great role in the earliest researches of Newton, concerning which we are only very meagerly informed. Fortunately the evolution of this detail can be surveyed rather accurately. Newton probably was occupied in the years 1665 and 1666 with the quadrature of the hyperbolic segment.[19]

By a purely numerical attack which corresponds to the series of

Mercator and Wallis he arrived at the surface segments. However, from the beginning on he calculated this from half the sum and difference, and was thus in possession of Gregory's results. He also knew the relation to logarithms.[20] In the *Analysis per æquationes numero terminorum infinitas*[21] and in the *Methodus fluxionum et serierum infinitarum cum ejusdem applicatione ad curvarum geometriam*[22] the presentation gradually becomes more general and in a letter of January 19, 1669, to Collins[23] there appears not only the series for the hyperbolic segment represented by

$$\log \frac{a+x}{a-x},$$

but also the remark that this series converges twice as rapidly as the original series of Mercator.

In the *Analysis per æquationes* the logarithmic series moreover is cleverly gotten by reversion by means of gradual approximation. In the first letter to Oldenburg for Leibniz of June 13, 1676[24], this reversion function is written in quite general form and in the second letter to Oldenburg for Leibniz of October 24, 1676[25] it is gotten out by a sort of comparison of powers. Moreover Newton gives in the *Methodus fluxionum* a process, not yet entirely mature, of finding the logarithm of a from the already known logarithms of $a \pm x$ and the given x.

IV. *The Methodological Expansion.* With these developments the formulation of the fundamental material, namely the setting up of the logarithmic series and its reversion, is completely closed. That which follows is refinements in details and methodological improvements. They were not immediately successful, indeed not until a full quarter of a century after the first discoveries; at a time therefore, when the new thoughts were no longer so strange and unusual.

Foremost is Edmund Halley (1656-1742) with his attempt to eliminate the hyperbolic surface,—that painful transition structure between the series and the logarithm.[26]

He explains the logarithmic series and its reversion from the binomial theorem for infinitely small *resp.* infinitely large exponents. In modern terms his process reduces to the change of limits

$$\log(1+x) = y = \lim_{n\to\infty} \frac{(1+x)^{1/n}-1}{1/n}$$

and

$$(1+x) = \lim_{n\to\infty} \left(1+\frac{y}{n}\right)^n$$

It is expressed however without the change of limits, only with $n=\infty$ and in very obscure form. Doubtless Halley first set $n=\infty$ in the binomial series, then established the connection with the logarithmic series and accordingly attempted to get it by reversion. But this was only very slightly successful; it remained in a changed dress with fine but obscure words.

A few years later Abraham de Moivre (1667-1754) explains in a short essay the attack on the undertermined coefficients in the transformation of series—at that time no longer much of which was new as to thought content.[27] In it the deduction of the logarithmic series is also touched on. It occurs thus:

If $1+z$ is a number and its logarithm is expressed by the series $az+bz^2+cz^3+\cdots$, then the logarithm $ay+by^2+cy^3+\cdots$ belongs to $1+y$. If now $1+z=(1+y)^n$, therefore

$$az+bz^2+cz^3+\cdots=nay+nby^2+ncy^3+\cdots,$$

then one can develop and insert $z=(1+y)^n-1$ according to the binomial theorem. Then the a,b,c etc. can be determined by comparison of coefficients with the exception of the first a, which remains arbitrary and characterizes the different kinds of logarithms. That is really remarkable enough; for the n which remained arbitrary falls completely out of the calculation. Unfortunately this is not expressly emphasized by de Moivre, although he doubtless knew it. Fundamentally de Moivre attacked the problem as it is done with a functional equation. The passage under consideration is certainly one of the earliest examples of it.

Twelve years later Roger Cotes (1682-1716) again takes up the same thought. He recognizes its deeper meaning and thinks it through clear to the end.[28]

The result is the new definition of the logarithm from the functional equation $f(a^x)=x\cdot f(a)$. Cotes doesn't yet have our functional signs. He replaces it by a designation which he knew basically since Mercator: $f(a)$ means for him the measure of the relationship a[29]. The functional equation is now solved by an infinitesimal method which is equivalent to our treatment by means of differential calculus. Hence there follows

$$f\left(\frac{c+x}{c-x}\right)=M\cdot\left(\frac{x}{c}+\frac{x^3}{3c^3}+\frac{x^5}{5c^5}+\cdots\right)$$

and the modulus M characterizes the various logarithmic systems which can be gotten in such a manner. By reversion of the logarithmic series Cotes on this occasion comes to a continued-fraction develop-

ment of e (doubtless based on direct dividing out of his result calculated to twelve decimal places) and finally gives the quadrature of the hyperbolic segment and the hyperbolic sector by means of logarithms. These calculations of his we can designate in modern style and very aptly as "logarithmic integration"; that is the essence of the *Logometria*. It is a new method—we would designate it as an example of the substitution method in the transformation of indefinite integrals—by means of which an abundance of contemporary but apparently mutually unrelated separate results could be explained from a single guiding viewpoint. Cotes himself provided for the best in this respect in the *Logometria* and the fragments of the *Harmonia mensurarum* which he completed; Smith added nothing new of his own.

If we look back then we recognize in this entire development from Mercator to Cotes a coherent line. The thought content becomes piecemeal richer and richer, the form better and better, more and more complete. Cotes stands as the last on the shoulders of all his predecessors. He really gives something finished and complete.

We must add that Leibniz in 1673 worked through Mercator's *Logarithmotechnia* and at least after 1676 was in possession of the reversion of the logarithmic series. However the exact details cannot be presented at the moment as long as the Leibniz edition is not completed; for it will certainly bring new material which remained hitherto inaccessible. Meanwhile, the activity of Leibniz and his school had no further influence on the development of the logarithmic series in England; hence it could be ignored without essential loss.

To the reader of today much in the conception and mode of expression of that time appears strange and unusual. Between us and the mathematicians of the late seventeenth century stands Leonhard Euler (1707-1783). He is the real founder of our modern conception. However non-rigorous he may be in details: he ends and conquers the previous epoch of direct geometric infinitesimal considerations and introduces the period of mathematical analysis according to form and content. Whatever was written after him on the logarithmic series is necessarily based no longer on the already obscured predecessors in the receding mathematical Renaissance, but on Euler's *Introductio in analysin infinitorum*, Lausanne 1748, in which the entire seventh chapter treats of logarithms.

NOTES

[1] *Hypothesis Astronomica nova, ejusque cum Observationibus consensus*, London 1664, extended in the *Institutiones astronomicæ*. . ., London 1676. Here the geocentric and the heliocentric system is very thoroughly presented.

[2] *Euclidis Elementa Geometrica, novo ordine ac methodo fere demonstrata*, London 1666. The second edition (London 1678) is augmented by an *Introductio brevis in Geometriam*. This was included by Johann Andreas Schmid in his Latin editions (Jena 1684 and 1692) of the *Elémens de géométrie* of the Jesuit father Ignace Gaston Pardies (French first edition, Paris, 1671), which follow the third French edition (Paris, 1678). Mercator according to the entire set-up must be regarded as a very important precursor of Pardies. The latter's work is by no means as independent in structure and details as was formerly believed.

[3] *Problemata quædam, ad promotionem scientiæ navigatoriæ facientia*. Philosophical Transactions 2 (1666), pp. 161-163. Here Mercator promises among other things the proof of Henry Bond's assertion (expressed in Richard Norwood's *Epitome of Navigation*, 1645) that the map meridian of a Mercator projection (according to Gerhard Mercator, *Weltkarte*, Duisburg 1569) can be determined by a logarithmic tangent of the earth's meridian. But Gregory anticipated him with the publication of the proof (in the *Exercitationes geometricæ*, London 1668).

[4] *Logarithmo-technia; sive methodus construendi logarithmos nova, accurata, et facilis; scripto antehac communicata, anno sc.* 1667, *nonis Augusti. Cui nunc accedit vera quadratura hyperbolæ, et inventio summæ logarithmorum*, London 1668. Included is a memoir by Michel Angioli Ricci on extreme values, of secondary importance to us. The *Logarithmotechnia* was reprinted at London in 1674 and seems also to have been edited in an English translation by John Collins. A rather uncritical reprint is in Francis Masères, *Scriptores logarithmici* I, London 1791, pp. 169-196. On the more accurate details for the following cf. J. E. Hofmann, *Nicolaus Mercators Logarithmotechnia* (1668), Deutsche Mathematik 3 (1938), pp. 446-466.

[5] This first publication is hitherto unlocated.

[6] The introduction of the *ratiunculæ* (unfortunately forgotten today) had found firm footing after the *Arithmetica logarithmica sive logarithmorum chiliadis triginta*, London 1624, by Henry Briggs.

[7] If we set, *e. g.*, $1.005=g$, then Mercator forms g^2, g^4, g^8, g^{16}, g^{32}, g^{64}, g^{128}, g^{256}, and finds that $g^{512}>10$. Hence he reckons further $g^{256+128}$, g^{384+64}, $g^{448+32}>10$; $g^{448+16}>10$; g^{448+8}, g^{456+4}, $g^{460+2}>10$; $g^{460+1}<10$. From $g^{461}=9.165\cdot 774$ and $g^{462}=10.015\cdot 603$ he finds $g^{461.6868}\backsim 10$; therefore the number of *ratiunculæ* pertaining to g is ten million: $461.6868=21'6597$. Consequently the common logarithm of $g=0.0021\cdot 6597$ (instead of $0.0021\cdot 6606$).

[8] *moles*. Occasionally the designation *mensura rationis* also occurs, which later gains a new and definite meaning in Cotes.

[9] From the detailed explanations it seems to follow that Mercator had not studied Cavalieri's original works (Bologna 1635, 1645 and 1653) but had learned from lectures heard in Rostock, Copenhagen and Danzig.

[10] Phil. Trans. 3 (1668) pp. 753-759; reprinted in Masères[4] I, pp. 219-226.

[11] Phil. Trans. 3 (1668) pp. 759-764; reprinted in Masères[4] I, pp. 227-232.

[12] The details of this explanation are evident from the reprint in the *Oeuvres complètes de Christian Huygens* VI (Correspondance 1666-1669), The Hague, 1895.

[13] Here it has to do with the paper *N. Mercatoris quadratura hyperboles geometrice demonstrata. Exercitationes geometricæ* pp. 9-13, reprinted in Masères[4] II, London 1791, pp. 2-5. A more detailed account appeared in Deutsche Mathematik 3 (1938), pp. 598-605: J. E. Hofmann, *Weiterbildung der logarithmischen Reihe Mercators in England*, I.

[15] This is presented in the *Opus geometricum quadraturæ circuli et sectionum coni*, Antwerp, 1647.

[16] The essay was partially completed even in 1644, but did not receive its final form until 1657. The results became known from Fermat's letters, a copy lay in Paris with Pierre de Carvaci. Perhaps Gregory looked into them there. The paper was first printed in the *Varia Opera P. de Fermat*, edited by his son Samuel Fermat, Toulouse 1679, thus long after Gregory's death.

[17] The letter is dated April 9, 1672 and addressed to the weights and measures officer Michæl Dary, who rated as a good algebraist and was counted among the more intimate friends of John Collins. Unfortunately the original is lost; we must rely on a copy by Collins, in which the nomenclature was probably altered from that of the original. We find the letter in Stephen Jordan Rigaud, *Correspondence of Scientific*

Men of the XVIIth Century, Oxford 1841, p. 240. ...Dary passed the logarithmic series on to a certain Euclid Speidell who published in 1668 a scientifically unimportant *Logarithmotechnia* (reprinted in Masères[4] VI, London 1807, pp. 43-75). There the natural logarithms of the first primes are calculated to 25 places, on the basis of the hyperbola, after many sorts of formalities.

[18] The same subdominant appears later again—rather certainly independent of Gregory—in a letter of George Anderson on 16/27 September, 1740, to William Jones, printed in Rigaud[17] I, Oxford 1841, pp. 360-362.

[19] We rely here on Newton's explanations in his second letter to Oldenburg for Leibniz of October 24, 1676, which was first printed *in toto* in the ***Commercium Epistolicum J. Collins et aliorum de analysi promota*** (editions: 1712-13, 1722, 1725). I refer to the critical edition of J.-B. Biot and Fr. Lefort (Paris, 1856) in which the textual variants of the separate editions are expressly characterized.

[20] Probably he learned it through his teacher Isaac Barrow, who even at that time knew that the hyperbolic segment is proportional to the logarithm of the quotient of the including ordinates.

[21] Begun about 1665, finished in 1669 and in Collins as copy. Excerpts printed in John Wallis, ***Treatise of Algebra, both historical and practical***, London 1685 (extant as manuscript after 1676), Latin in the ***Opera mathematica*** of John Wallis, II, Oxford 1693-1695. The copy by Collins was first edited by W. Jones, London 1711, Newton's original in the ***Commercium Epistolicum***.[19]

[22] Written about 1670, again worked over supposedly about 1673, first printed by John Colson (London 1736).

[23] Rigaud[17] I, pp. 285-286.

[24] ***Commercium Epistolicum***,[19] pp. 108-109.

[25] ***Commercium Epistolicum***,[19] pp. 142.

[26] ***A most compendious and facile Method for constructing the Logarithms, exemplified and demonstrated from the Nature of Numbers, without any regard to the Hyperbola, with a speedy Method for finding the Number from the Logarithm given***, Phil. Trans. 19 (1695), pp. 58-67. A reprint with several non-essential alterations is in Masères[4] II, London 1791, pp. 84-91. Cf. further the more detailed presentation in Jos. E. Hofmann, ***Weiterbildung der logarithmischen Reihe Mercators in England***, III, (Deutsche Mathematik 4, 1939).

[27] ***A Method of extracting the Root of an Infinite Equation***, Phil. Trans. 20 (1698) pp. 190-192. Cf. also the presentation in Hofmann.[26]

[28] ***Logometria***, Phil. Trans. 29 (1714) pp. 5-45 The essay was already finished by the end of 1711. An approximately line-by-line exact reprint is found in the posthumous work ***Harmonia mensurarum, sive Analysis et Synthesis per rationum et angulorum mensuras promotæ; accedunt alia Opuscula mathematica***, Cambridge 1722, pp. 1-41. This wonderfully gotten up volume was edited by Robert Smith, the cousin and successor of Cotes.

[29] We moderns think analytically, for us the analytical concept of number is the causal. Cotes still thinks of everything in dimensional magnitudes, here therefore in the unnamed proportion. He calls his measure of proportion ***mensura rationis***. That corresponds to the λόγος ἀριθμός.

Mathematical World News

Edited by
L. J. Adams

Professor Edgar E. DeCou, head of the department of mathematics at the University of Oregon, was made Professor Emeritus on July 1, 1939, after thirty-seven years of service. Professor DeCou will continue part-time teaching. Professor DeCou has served the cause of mathematics well in the Pacific Northwest. Associate Professor Andrew F. Moursund, B. S. and M. S. (Texas) and Ph.D. (Brown), succeeds Professor DeCou as head of the department. As state representative of the National Council of Teachers of Mathematics, Professor DeCou recently presented the certificate of affiliation of the Portland Council of Teachers of Mathematics with the National Council. The importance of the achievement of Professor DeCou in effecting this affiliation can be appreciated when it is understood that the Portland Council serves one-third of the population of the State of Oregon.

Changes in rank and appointments at the University of Oregon include the following: Kenneth S. Ghent, B. A. and M. A. (McMaster University) and Ph.D. (Chicago) is advanced from instructor to assistant professor. Thurman S. Peterson, B. S. (California Institute of Technology) and M. S. and Ph.D (Ohio State) is advanced from instructor to assistant professor. Carl F. Cossack, B. S. and M. S. (University of California at Los Angeles) and Ph.D (Michigan) becomes instructor in mathematics.

The following have been appointed instructors at Harvard University for the academic year 1939-40: M. H. Heins, A. D. Hestenes, D. T. McClay, E. N. Nilson, A. Spitzbart, P. M. Whitman.

Professor J. H. Van Vleck of Harvard University gave a series of lectures at the Institute Henri Poincare at the University of Paris last May. Professor Van Vleck was also scheduled to take part in a symposium on magnetism at Strasbourg.

Professor G. D. Birkhoff of Harvard University was appointed Exchange Professor to France for the second half of the academic year 1939-40.

The following have been appointed Benjamin Pierce Instructors at Harvard University for the current academic year: Dr. Leon Alaoglu, Dr. Donald T. Perkins and Dr. B. J. Pettis.

Professor J. N. Michie, head of the department of mathematics at Texas Technological College, announces the following new members of the mathematics department for the 1939-40 session. Dr. Emmett Hazelwood of Mansfield, Pennsylvania will become assistant professor of mathematics. He has a B. S. degree from West Texas State Teachers College and M. A. and Ph.D. degrees from Cornell University. Dr. Hazelwood has served as instructor at West Texas State Teachers College and New Mexico A. and M., and was head of the department at State Teachers College, in Mansfield, Pennsylvania during the session 1938-39. Dr. Loyal F. Ollman of Shawno, Wisconsin will join the mathematics department with the rank of instructor. Dr. Ollman received a B. A. degree from Ripon College, M. A. degree from the University of Wisconsin and Ph.D. degree last June from the University of Michigan. He served as mathematics assistant at the University of Michigan from 1936 to 1939. Dr. Raymond K. Wakerling of Berkeley, California will join the department with the rank of instructor. He has B. A. and Ph.D. degrees from the University of California. He has served as teaching assistant in mathematics in the University of California from 1936 to 1939. Mr. E. R. Heineman of the Texas Technological College has been promoted from an assistant professorship to an associate professorship. Dr. Earl L. Thompson of the Texas Technological College has been promoted from an associate professorship to a full professorship.

Also from Professor Michie comes the following news items. Instructor Miller studied at the University of Michigan during the summer session, with Drs. Craig and Copeland. Instructor of Missouri during the summer session, Professor Heineman studied at the University of Texas with Dr. Moore in point set theory and with Professor Vandiver in number theory and finite groups.

The Southern Intercollegiate Mathematics Association held its sixth annual meeting in Jackson, Mississippi at Millsaps College on May 13, 1939. The schools having won the right to participate in the finals by virtue of winning in the preliminary contests in their respective regions were: Mississippi Woman's College, Centenary College, Southern Methodist University and North Texas State Teachers College. Southern Methodist University was awarded the S. I. M. A. Cup for having the highest average in the final examinations. The following students won the right to wear the S. I. M. A. Key as a result

of their grades. Preliminary Examination winners: Merle Mitchell (S. M. U) in Algebra; Cleo White (Mississippi Woman's College) in Trigonometry; Mary Emma Fancher (Mississippi Woman's College) in Analytics; Nina Pearl Byrd (Mississippi Woman's College) in Calculus; Louise Tate (Mississippi Woman's College) in the Comprehensive. Final examination winners: Merle Mitchell (S. M. U.) in Algebra; Julia Smith (S. M. U.) in Trigonometry; Wilbur Teubner (S. M. U.) in the Comprehensive; William Tittle (North Texas State Teachers College) in Calculus; William E. Steger (Centenary College) in Analytics. The following member professors attended: Mouzon, Huff, Palmquist, Wright (Southern Methodist University); Herdin, Banks (Centenary College); Mitchell, Van Hook (Millsaps College); Hanson, Cooke (North Texas State Teachers College); Brown (Mississippi Woman's College); Dearman (Mississippi State Teachers College); Tate (MacMurray College).

The *Boletin Matematico* of Buenos Aires is now the unique Spanish publication not only in Latin America but all over the world.

The American Mathematical Society met at the University of Wisconsin in Madison, Wisconsin on September 5-8, 1939. The Colloquium Lectures on *Structure of Algebras*, were given by Professor A. A. Albert, of the University of Chicago. The Colloquium Lectures on *Convex Bodies*, were given by Professor M. H. Stone, of Harvard University.

The Mathematical Association of America, met at Madison, Wisconsin, on September 4-7, 1939. Addresses scheduled were:

1. *Algebra for the undergraduate*, Professor Marie J. Weiss, Sophie Newcomb College.
2. *Over and under functions as related to differential equations*, Professor W. M. Whyburn, University of California at Los Angeles.
3. *Soap films and the calculus of variations*, Professor W. T. Reid, University of Chicago.
4. *The role of isomorphism in mathematics and its applications*, by Professor A. J. Kempner, University of Colorado, retiring president of the Association.
5. *Report of the Committee on Tests*, Professor E. Q. Chittenden, University of Iowa.

Hirschwaldsche Buchhandlung, Berlin NW 7 Unter den Linden 60, announces *Grundlagen der Mathematik*, by Dr. D. Hilbert and Dr. P. Bernays. (Second volume).

The National Council of Teachers of Mathematics met with the National Education Association on July 3, 4, 5, 1939 in San Francisco, California. The general theme was *Teaching Mathematics to Meet Social Needs*. The program included many excellent addresses and panel discussions.

On the occasion of the sixty-fifth anniversary of M. Élie Cartan, a group of friends, colleagues, students and professors of many nations propose to celebrate the scientific jubilee of the eminent geometer by publishing a volume of his works. Also, a volume of the *Journal de Mathématiques* will be dedicated to him.

The University of Illinois mathematics department recently gave a banquet in honor of Professor Arnold Emch who is retiring from teaching at the close of the 1939 summer session.

The University of Kansas conferred the Ph.D. degree on Gilbert Ulmer at the 1939 commencement, and has appointed him an assistant professor in the department. His doctoral thesis *Teaching Geometry to cultivate reflective thinking; an experimental study with 1239 high school students*, has been declared a valuable study.

M. A. degrees conferred at University of Kansas included:

1. Charles Albert Burgtorf. (Thesis, *An analytical treatment of isogonal conjugate curves.*)
2. Catherine C. DeTar. (Thesis, *The Inscribed, circumscribed, escribed and nine-point circles of certain families of triangles.*)

The Missouri Academy of Science closed its 5th annual meeting (three days session) in Springfield (Missouri) on April 29, 1939. Miss Emma Gibson read a paper by her teacher, Dr. B. F. Finkel of Drury College, on determining the derivatives of a^x and arc sin x without changing to other forms. It is entitled, *An ab initio derivation of the derivatives of a^x and arc sin x.* Professor Finkel also gave a paper on the development of mathematics in Missouri.

Problem Department

Edited by
ROBERT C. YATES and EMORY P. STARKE

This department solicits the proposal and solution of problems by its readers, whether subscribers or not. Problems leading to new results and opening new fields of interest are especially desired and, naturally, will be given preference over those to be found in ordinary textbooks. The contributor is asked to supply with his proposals any information that will assist the editors. It is desirable that manuscript be typewritten with double spacing. Send all communications to ROBERT C. YATES, Mathematics, University, Louisiana.

SOLUTIONS

No. 105. Proposed by *Raymond A. Lyttleton*, Princeton University.

If an ellipse is inscribed in a triangle with its center at the circumcenter, then the altitudes are normals.

Solution by *F. C. Gentry*, Louisiana Polytechnic Institute.

Assume a system of normal trilinear coordinates in which the homogeneous coordinates (x_1, x_2, x_3) of a point x are proportional to the distances of x from the sides of the reference triangle. Then the equation of a conic touching the sides of the triangle is of the form

$$r^2x_1^2+s^2x_2^2+t^2x_3^2-2stx_2x_3-2rtx_1x_3-2rsx_1x_2=0.$$

The center of this conic is the pole of the line at infinity:

$$ax_1+bx_2+cx_3=0,$$

where a, b, c are the lengths of the sides of the triangle of reference. Hence the coordinates of the center satisfy the system of equations:

$$r^2x_1-rtx_2-rsx_3=\rho a$$

$$-rsx_1+s^2x_2-rtx_3=\rho b$$

$$-rtx_1-stx_2+t^2x_3=\rho c,$$

ρ a constant, and the homogeneous coordinates of the center are $(tb+sc,\ ta+rc,\ sa+rb)$. But the coordinates of the circumcenter 0 of the triangle are $(\cos A, \cos B, \cos C)$ where A, B, C are the angles of

the triangle. From the condition that these two sets of coordinates are proportional we have $r:s:t=a^2 \cos B \cos C : b^2 \cos A \cos C : c^2 \cos A \cos B$. The altitude h_a of the triangle has the equation $x_2 \cos B - x_3 \cos C = 0$. It meets the conic $a^4 \cos^2 B \cos^2 C\, x_1^2 + b^4 \cos^2 A \cos^2 C\, x_2^2 + c^4 \cos^2 A \cos^2 B\, x_3^2 - 2b^2c^2 \cos^2 A \cos B \cos C\, x_2x_3 - 2a^2c^2 \cos A \cos^2 B \cos C\, x_1x_3 - 2a^2b^2 \cos A \cos B \cos^2 C\, x_1x_2 = 0$ in the two points whose coordinates are $((b^2-c^2) \cos A, a^3 \cos B \cos^2 C, a^3 \cos^2 B \cos C)$ and $(\cos A, \cos B \cos^2 C, \cos^2 B \cos C)$. The tangent to the conic at the latter of these points is $a \cos B \cos C\, x_1 - b \cos A\, x_2 - c \cos A\, x_3 = 0$. It is perpendicular to h_a since $-b \cos A \cos B + c \cos A \cos C - (b \cos A \cos C - c \cos A \cos B) \cos A = 0$. Hence h_a, and similarly h_b and h_c, is normal to the conic. It will be noted that the theorem is true for any conic touching the sides of the triangle and having its center at the circumcenter.

No. 224. Proposed by *Dewey C. Duncan*, Compton Junior College, California.

If the external bisectors of the base angles of a scalene triangle are equal, prove that

(a) If the two sides are given, the triangle is uniquely determined but *not* constructible by Euclidean means.

(b) If the base b and one side a $(b>a)$ are given, the triangle is uniquely determined and *is* constructible by Euclidean means.

Solution by *C. W. Trigg*, Los Angeles City College.

The square of an external bisector, T_A, of angle A of a triangle is equal to

$$bc\left[\frac{a^2}{(b-c)^2}-1\right].$$

So if $T_C = T_A$, then

$$ab\left[\frac{c^2}{(b-a)^2}-1\right] = bc\left[\frac{a^2}{(b-c)^2}-1\right].$$

This expression simplifies to

$$b(a-b+c)(c-a)[b^3-(a+c)b^2+3acb-ac(a+c)]=0.$$

The first two factors cannot be zero, so either $(c-a)=0$ and the triangle is isosceles, or

$$b^3-(a+c)b^2+3acb-ac(a+c)=0. \tag{1}$$

Consider that the sides are measured in a scale such that a and c have no common factor greater that unity. If this equation has a rational root it is either a, c, or $(a+c)$. The first two suppositions require that $a=b=c$, and the last requires that $b=a+c$, which is impossible.

(a) Hence if the triangle is scalene with base b, and a and c are given, then (1) has no rational root. The discriminant of this cubic in b is $4ac\,[9ac(a+c)^2-(a+c)^4-27a^2c^2]$ which may be written in the form

$$-4ac\{[(a+c)^2-5ac]^2+ac(a+c)^2+2a^2c^2\}.$$

Since this discriminant is negative, the equation has a single real root and two conjugate imaginary roots. But it is not possible to construct with ruler and compasses a line whose length is a root of a cubic equation with rational coefficients having no rational root. Therefore, b, and hence the triangle, is uniquely determined, but *not* constructible by Euclidean means.

(b) b and one side a are given, then (1) may be rewritten as a quadratic in c, that is,

$$ac^2+(a^2-3ab+b^2)c-(b-a)b^2=0.$$

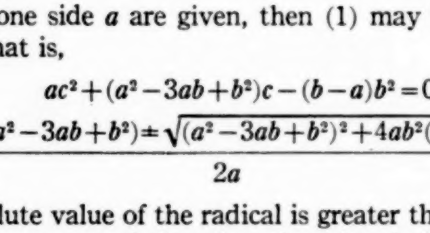
Whence, $$c=\frac{-(a^2-3ab+b^2)\pm\sqrt{(a^2-3ab+b^2)^2+4ab^2(b-a)}}{2a}.$$

If $b>a$, the absolute value of the radical is greater than the other part of the numerator, so c, and therefore the triangle is uniquely determined. Since c is derived from a and b by a finite number of rational operations and extraction of real square roots, it and the triangle are constructible by Euclidean means.

No. 226. Proposed by *V. Thébault*, Le Mans, France.

Given any four spheres, show that the lines joining the internal centers of similitude of each pair of spheres with the other meet in a point.

Solution by *C. W. Trigg*, Los Angeles City College.



Indicate the centers of the spheres by A_i and their radii by $r_i(i=1, 2, 3, 4)$, and denote the centers of similitude by B_{jk}, where j and k are the subscripts of the associated spheres. In the general case, the centers of the spheres are the vertices of a tetrahedron.

It is well known that the internal center of similitude of two spheres divides the line of centers proportionally to their radii. So

$$A_1B_{12} : B_{12}A_2 :: r_1 : r_2, \quad A_2B_{23} : B_{23}A_3 :: r_2 : r_3, \text{ etc.}$$

Hence

$$\frac{A_1B_{12}}{B_{12}A_2}\cdot\frac{A_2B_{23}}{B_{23}A_3}\cdot\frac{A_3B_{34}}{B_{34}A_4}\cdot\frac{A_4B_{41}}{B_{41}A_1}=\frac{r_1}{r_2}\cdot\frac{r_2}{r_3}\cdot\frac{r_3}{r_4}\cdot\frac{r_4}{r_1}=1.$$

Therefore, by the converse of Carnot's theorem (Altshiller-Court, *Modern Pure Solid Geometry*, p. 111) the points B_{12}, B_{23}, B_{34} and B_{41} are coplanar. So $B_{41}B_{23}$ and $B_{12}B_{34}$ intersect in some point M.

In like manner it may be shown that $B_{24}B_{31}$ and $B_{12}B_{34}$ are in the plane $y(B_{12}B_{24}B_{34}B_{31})$, and that $B_{31}B_{24}$ and $B_{41}B_{23}$ are in the plane $z(B_{24}B_{41}B_{31}B_{23})$. Now M is on $B_{41}B_{23}$ so is in the plane z, and M also lies on $B_{12}B_{34}$ so is in the plane y. Hence M lies on the intersection of these two planes, which is $B_{31}B_{24}$. Therefore the three joins of the centers of similitudes of the opposite pairs of spheres pass through a point.

Since any situation in which the centers of the spheres are coplanar may be duplicated by projecting the vertices of a tetrahedron upon a plane (followed possibly by projection upon a line) in which process the ratios would be invariant, any special or degenerate case would still satisfy the theorem.

Also solved by the *Proposer* who remarks that the question generalizes the particular case of four spheres tangent in pairs which has been given by N. A. Court in *Mathesis*, 1933—228.

Late Solution: No. 268 by *C. W. Trigg*.

No. 277. Proposed by *C. W. Trigg*, Los Angeles City College.

Find a square number of the form *abcdefgh* with $ab=gh$ and such that *cdef* is a square number. A solution without the aid of tables is desired.

Solution by *Johannes Mahrenholz*, Gottbus, Germany.

Put p^2 for *abcdefgh* and q^2 for *cdef*. Then we have

$$p^2-(10q)^2=(10a+b)(10^6+1)$$

or

$$(p-10q)(p+10q)=(10a+b)\cdot 101\cdot 9901.$$

If we take $p+10q=9901$* and $p-10q=101(10a+b)$, we have $20q=9901-101(10a+b)$ from which it is easy to see that a is even and $b=1$. Also, since $q \leqq 99$, we must have

$$10a+b=(9901-20q)/101>78.$$

Hence $10a+b=81$ and we have the unique solution, $q=86$ and $p^2=(9041)^2=81739681$.

Also solved by the *Proposer*.

No. 278. Proposed by *V. Thébault*, Le Mans, France.

In what system of numeration does there exist a square of four digits having the form $aabb=(cd)^2$, with $a+b=c+d$?

Solution by *C. W. Trigg*, Los Angeles City College.

Any number is congruent to the sum of its digits modulo $r-1$, where r is the base of the scale of numeration. Hence $(cd)^2 \equiv (c+d)^2$ and $aabb \equiv 2(a+b) \bmod r-1$. These results, with $a+b=c+d$, give

$$(1) \qquad (a+b)(a+b-2) \equiv 0 \bmod r-1$$

The given data may be expressed as $(ar^2+b)(r+1)=(cr+d)^2$ or

$$(2) \qquad [a(r^2-1)+(a+b)](r+1)=[c(r-1)+(c+d)]^2.$$

(1) is satisfied if we suppose $a+b=r+1=c+d$. Then (2) gives $[a(r-1)+1](r+1)^2=(cr+d)^2$, so that $r+1$ is a factor of $cr+d$. That is, $(cr+d)/(r+1)=c+(d-c)/(r+1)$ is an integer; but $|d-c|<r+1$, whence $d-c=0$ and $c=d=(r+1)/2$. Thus $a(r-1)+1=c^2=(r+1)^2/4$. When this is solved for r, we find $r=4a-3$. It follows that $b=3a-2$, $c=d=2a-1$. Thus we have a one-parameter solution for $a>1$. The first few specific solutions are:

$$(33)^2=2244,\ (55)^2=3377,\ (77)^2=44\ \overline{10}\ \overline{10},\ \text{etc.}$$

where the scales have base 5, 9, 13, etc., respectively.

Again, (1) is satisfied if we suppose $a+b=r-1=c+d$, with r even.† Then, from (2), we have

$$(3) \qquad [a(r+1)+1](r+1)=(r-1)(c+1)^2,$$

*Since p^2 cannot have terminal digits 97, 98 or 99, neither can $p^2-(10q)^2$, nor can $10a+b$. Thus $10a+b \leqq 96$. But then $101(10a+b) \leqq 9696$, so that 9901 must be a divisor of the larger factor, $p+10q$. Furthermore, $p<10^4$ and $q<10^2$ imply $p+10q<11000$. —ED.

†It is not difficult to show that r odd yields no solution.

with $r+1$ and $r-1$ relatively prime. Then $r+1$ is a factor of $(c+1)^2$ but is relatively prime to $a(r+1)+1$. Thus $r+1$ is the square of an odd number. (3) may be put

$$(c+1)^2/(r+1) = [a(r+1)+1]/(r-1) = a+(2a+1)/(r-1).$$

Since the last member is integral and a cannot exceed $r-1$, we must have $2a+1=r-1$. ($2a+1=2r-2$ is impossible since $2a+1$ is odd.) Put $r+1=(2x+1)^2$, where $x>0$. Then $r=4x^2+4x$, $a=2x^2+2x-1$, $b=2x^2+2x$, $c=-1+(2x+1)(2x^2+2x)^{1/2}$, and $d=r-1-c$. In order that c may be an integer, b must be a square number, say n^2. Then the equation for b becomes $2n^2=(2x+1)^2-1$. Each of the infinite number of solutions of this Pellian equation yields a solution to our problem. The first specific solutions thus secured are: $x=1$, $3344=(52)^2$, scale of 8; and $x=8$, $\overline{143}\ \overline{143}\ \overline{144}\ \overline{144}=(\overline{203}\ \overline{84})^2$, scale of 288.

For other problems dealing with the relationship $aabb=(cd)^2$ see the *Amercian Mathematical Monthly*, February and March, 1939, pp. 107, 111 and 168.

Also solved by *Johannes Mahrenholz* and the *Proposer*.

Editor's Note. It seems hopeless to obtain a general solution. One finds easily solutions which do not come under either case treated above, for example: $99\ \overline{23}\ \overline{23}=(\overline{21}\ \overline{11})^2$ scale of 49 employs the relation $a+b=c+d=32$.

No. 281. Proposed by *Robert C. Yates*, Louisiana State University.

Given a circle, C, together with its center, and a line, L, both in a plane. With the straightedge alone construct a perpendicular to a given point, P, of the line.

Solution by *J. Rosenbaum*, Bloomfield, Connecticut.

Before giving the construction called for, the corresponding constructions will be shown, with the stated restriction as to tools, for the following special cases:

Case 1. *When L passes through center, O, of C, and P is not on L and not on C.*

Construction. Draw PA and PB, where A and B are the intersections of L with C. If A' and B' are the other intersections of PA and PB with C, then AB' and BA' are two altitudes of triangle PAB, and hence their intersection Q, together with P determine the altitude to AB. So that PQ is the required perpendicular.

Case 2. *When P is center O, and L does not pass through O.*

Construction. Take two arbitrary points, A and B, on L (but not on C). In triangle OAB draw the two altitudes from A and B (by case 1). Their intersection Q, together with O determine the required perpendicular.

Case 3. *When L is a secant other than a diameter, P is not on C, and OP is not parallel to L.*

Construction. Let A and B be the intersections of L with C. Draw diameter AD, and then draw BD (note that BD is perpendicular to L). Now draw line M through O perpendicular to BD (by case 2). Since M and L are parallel, the perpendicular to M drawn through P (by case 1), is the one required.

Case 4. *When L is arbitrary, and P is not on L.*

Construction. Through P draw two secants so as to cut L in A and B respectively, and such that A and B shall be not on C. Next, draw the altitudes from A and B in triangle PAB (by case 3). Their intersection, Q, together with P determine the required perpendicular.

The construction for the problem is now as follows:

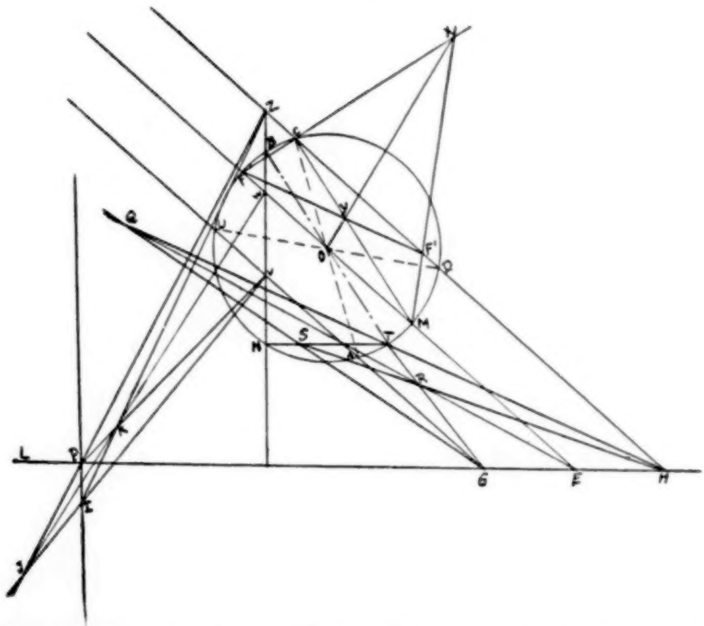
When P is not on C, and L is not a diameter: Draw diameter, D, perpendicular to L (case 2). Draw a secant, S, perpendicular to D (case 1). Then the perpendicular to S drawn through P (case 3) is the required line.

When P is not on C, and L is a diameter: Draw a secant, S, perpendicular to L (case 1). Draw a secant, T, perpendicular to S (case 3). Then the perpendicular to T drawn through P (case 3) is the required line.

When P is on C, and L is not a diameter: If the perpendicular to L drawn through O (case 2) passes through P then this perpendicular is the required line. Otherwise, L must be a secant. The two points common to L and C are the point P, and a second point Q. Here, draw the diameter QR, then RP is the required line.

When P is on C, and L is a diameter: Draw a secant, S, perpendicular to L (case 1). Draw line M perpendicular to S (case 3). Finally, the perpendicular to M drawn through P (by case 4) is the required line.

Solution by *C. L. Steininger*, Rossford, Ohio.



The solution to this problem depends upon the fundamental construction of drawing a line parallel to a given line on which three points are given such that one is equidistant from the other two.

In the figure construct any diameter FOM cutting line L at E. Draw any secant FC and locate on it any point X. Draw OX, MX and CM cutting OX at Y. Draw FY intersecting MX at F'. Draw CF' cutting L at H and the circle at D. Now CH is parallel to FE. Draw diameters, DOU, COA. Draw UA intersecting L at G. Now UG is parallel to FE and furthermore GE equals EH.

Choose any point T on the circle. Draw HT, on which locate some point Q. Draw EQ, GQ, and GT intersecting EQ at R. Draw HR cutting GQ at S. Draw TS cutting the circle at N. Draw diameter TOB. Draw BN cutting the parallels, UG, FE and CH at V, W, and Z respectively. NT is parallel to L and BN is perpendicular to L.

There remains to draw a line through P parallel to BN. This is accomplished by drawing ZP to any point such as J. Draw WJ, VJ and VP cutting WJ at K. Draw ZK intersecting VJ at I. Draw PI the required line.

This problem and proof appears in *Procédés originaux de constructions géometriques* by E. Fourrey, Paris, 1924. See also Steiner's *Geometrischen Constructionen*, 1913, pp. 52-55.

Also solved by *L. M. Kelly, D. L. MacKay, Johannes Mahrenholz, C. W. Trigg*, and the *Proposer*.

PROPOSALS

No. 310. Proposed by *H. A. Simmons*, Northwestern University.

Derive the formula

$$\tan(x/2)=\sin x/(1+\cos x),$$

without the use of any irrational expression.

No. 311. Proposed by *Leo F. Epstein*, Massachusetts Institute of Technology.

Prove the relation

$$\sum_{m=1}^{\infty}\frac{1}{m!}\sum_{t=1}^{\infty}\frac{(\ln t)^m}{t!}=1.$$

No. 312. Proposed by *Walter B. Clarke*, San Jose, California.

Through the vertices of a regular pentagon draw five lines forming an irregular pentagon. Erase the former and obtain it again by Euclidean construction. Under what conditions will an arbitrary pentagon circumscribe a regular pentagon?

No. 313. Proposed by *C. W. Trigg*, Los Angeles City College.

I. Find a number with three like middle digits whose square is a permutation of the ten digits.

II. Using only two consecutive digits form a five-digit number whose square is a permutation of the ten digits.

Each has a unique solution.

No. 314. Proposed by *D. L. MacKay*, Evander Childs High School, New York.

Given points A and B outside a given circle. Determine a point X on the circle such that AX and BX will have equal chords within the circle.

No. 315. Proposed by *H. S. Grant*, Rutgers University.

$F(x,y)=0$ and $F(\rho\cos\Theta,\ \rho\sin\Theta)=0$ when solved for y, x, and ρ

explicitly define single-valued, continuous functions of x, y, and Θ, respectively. Consider

$$A_1=\left|\int_{x_1}^{x_2} ydx\right|; \qquad A_2=\left|\int_{y_1}^{y_2} xdy\right|; \qquad A_3=\left|\int_{\alpha}^{\beta} \rho^2 d\Theta\right|;$$

where (x_1,y_1) and (x_2,y_2) are the rectangular coordinates of any two points on the curve and $\alpha=\text{arc }\tan(y_1/x_1)$, $\beta=\text{arc }\tan(y_2/x_2)$. For what functions is (1) $A_1=A_2$; (2) $A_1=A_3$; (3) $A_2=A_3$; (4) $A_1=A_2=A_3$?

No. 316. Proposed by *H. T. R. Aude*, Colgate University.

If two rectangles *ABCD* and *EFGH* are located in the plane, and if the midpoints of the line segments *AE*, *BF*, *CG*, and *DH* are denoted respectively, *I*, *J*, *K*, and *L*; show that if the figure *IJKL* is a quadrilateral it is a parallelogram. For two rectangles in fixed positions how many parallelograms may thus be formed?

No. 317. Proposed by *Nathan Altshiller-Court*, University of Oklahoma.

Given three collinear points *A*, *B*, *C*. The perpendiculars *BP*, *CQ* are dropped from *B* and *C* upon a variable plane passing through *A*. Find the locus of the point $M=(BQ, CP)$

From *Wright's Mechanics:*

Swift of foot was Hiawatha:
He could shoot an arrow from him
And run forward with such fleetness
That the arrow fell behind him!

Strong of arm was Hiawatha:
He could shoot ten arrows upward,
Shoot them with such strength and swiftness
That the tenth had left the bowstring
Ere the first to earth had fallen.

If one second elapsed between the discharge of each of the arrows and Hiawatha shot at his greatest range, prove that he must have been able to run at the rate of 99 miles an hour.

Bibliography and Reviews

Edited by
H. A. Simmons

Mathematical Analysis for Economists. By R. G. D. Allen. The Macmillan Company, New York, 1939, xv+548 pages. Price $4.50.

During the last decade there has been a noticeable increase in the amount of mathematics used in the analysis of economic data and in the description of economic phenomena. It is interesting to note, however, that *mathematical economics* began more than a century ago with the publication of the now classical *Mathematical Principles of the Theory of Riches* by A. A. Cournot in 1837. In this book there were laid the foundations for a functional theory of money, prices, supply and demand, the production of goods, etc. On the other hand, the *statistical study* of the relationships posited by Cournot is largely a development of the present century. The merging of these two streams, *mathematical speculation* on the one side and *statistical formulation* on the other, is a phenomenon of recent years and this union is finding expression in the publications of the newly created field of *econometrics.*

It is very fortunate, therefore, that a book of the excellence of that of Mr. Allen should appear at this time. The book may be characterized as belonging to the tradition of Cournot, since no statistical evidence is introduced in its pages. But the author attempts to survey all that body of mathematics which has been found useful in formulating the problems of economics. This carries him a long distance since the book begins with the elements of our number system and terminates with some of the early theorems of the calculus of variations.

Any student who has mastered the details of the twenty chapters which comprise the book, and who has solved the plentiful supply of problems, will be well equipped for substantial application of mathematics not only in the field of economics, but also in other applied fields. He will have learned the elements of analytic geometry, of differential and integral calculus, of partial differentiation, the theory of maxima and minima of several variables, elementary differential equations, the nature and use of Taylor's series, determinants, linear equations, quadratic forms, and the calculus of variations.

The book is also an excellent introduction to the concepts useful in describing economic behavior. Illustrative examples and many

problems are chosen from this field. Thus under logarithms and exponents the author discusses problems of capital and interest and of the elasticity of demand. Under differentiation, he treats the general problem of average and marginal values and problems of monopoly, duopoly, etc. Under differential equations, he develops the theory of dynamic forms of demand and supply functions. The author also finds it necessary to introduce the theory of quadratic forms in order to treat adequately the demand for consumers' goods subject to budgetary restraint.

A number of problems in the calculus of variations are solved and application is made to the maximization of the profit function

$$u=\int_{t_0}^{t_1} [xp-\pi(x)]\, dt,$$

where x is the market demand, p the price, and $\pi(x)$ the cost of producing and marketing x units.

It is thus obvious that the new economics is imposing an unusual mathematical requirement upon the economist, who wishes to stay abreast of modern developments in his field. Similarly, the demand of economists for mathematical solutions of their problems is imposing a duty upon mathematicians to understand and to provide for this need. One is perhaps reminded in this connection of the bargain proposed to Alice by the Unicorn: "If you'll believe in me, I'll believe in you."

Northwestern University. H. T. DAVIS.

Calculus. By F. H. Miller. John Wiley and Sons, Inc., New York. xiv+419 pages. $3.00.

The author has succeeded in keeping the tone of this text book on a middle note; he has avoided the mechanical, formalistic approach, and he develops proofs with only so much rigor as he feels will be teachable. For example, Rolle's Theorem is omitted, and the Law of the Mean is made to depend upon a proof which is "apparent geometrically". Explanations and representative examples are numerous and well chosen. At all stages an attempt is made to show *why*, as well as *how*. (See, for example, the paragraph on the integrating factor of a linear differential equation of the first order.)

The material is that usually covered by a good introductory calculus, and is suitable for a 4 to 5 hour course for one year; the dy/dx notation is used from the start of differentiation, though other notations

are used occasionally. The definite integral is introduced before the indefinite integral, and by clever devices Duhamel's Theorem is avoided. Instead of presenting several pages of curves with equations appended, as is found in some texts, the author introduces a chapter on the drawing of algebraic curves, with emphasis on asymptotes and singular (*i. e.*, multiple) points. Thus the student is enabled, by use of differential calculus, to draw curves which are used later in integral calculus. The reduction-formula phase of integration is given very little space, many applications of integral calculus both to geometry and to mechanics are included.

The text closes with ample (for an introductory course) chapters on series and differential equations. These are followed by reference formulas, useful tables of logarithms, trigonometric functions and e^x, and a modest set of 100 integrals, and answers to problems. The type used is quite satisfactory, and there seem to be very few misprints or errors in the text.

To one who approves of the point of view of the author, only minor faults are apparent. To the reviewer (1) it seems desirable when a graph of a curve with given equation is drawn to indicate the scale on each axis (this is done only occasionally; see figures 1-3, 49-53, 107, 108); (2) the answers are too numerous (*e. g.*, when a series is to be found divergent or convergent it seems unwise to give the answer for every such problem) and are without sufficient regularity (for in many sets every problem has the answer furnished, for others only the odd problems, and for others still other schemes are used); (3) the problems, while very numerous and for the most part well chosen, are in some cases very difficult, and the teacher should not assign problems at random. This is particularly true of the sets on differentiation.

This book should be teachable to well prepared, good students; for others the text may furnish the *how* satisfactorily.

Northwestern University. FRANK EDWIN WOOD.

Trigonometry. By N. J. Lennes and A. S. Merrill. Harpers, N. Y. Revised Edition. xii – 243 pp. with five-place tables.

It has been said—with some justification—that all one needs in order to produce a new *Trigonometry*, is scissors and a glue pot. The Trigonometry under review is definitely *not* of this stripe. The authors have quite obviously made every effort to insure accuracy, readability and variety of exercises. They have succeeded admirably in making the subject breathe and live and have thus made it not only service-

able but also attractive to the beginner. The book contains a necessarily brief historical account of the growth of trigonometry; it is written in a true scientific spirit which is never satisfied with a mere chronology. The book seems especially suitable to a class interested in mathematics for its own sake more than in its prosaic applications.

The reviewer would deem himself derelict in his duty were he to fail to point out, what in his opinion are, defects in the book.

In the first place the computational part of the subject is emphasized at the expense of the so-called "analytical" part; in this it follows the modern trend of first learning how to solve triangles before the formulas necessary for their solution have become thoroughly familiar. There is no indication anywhere in the book as to how to use logarithms if negative numbers are involved. The proof of the formulas for the sine and cosine of the sum of two general angles would be greatly simplified if the functions of $n90° \pm A$ were studied beforehand. The use of ∞ follows traditional lines, even though the authors are careful to state that "an angle of 90°, *strictly speaking*, has no tangent" (p. 117). A little further on they upset the apple cart as follows: "It is sometimes said that $\tan 90° = \infty$. This must be taken to mean that as θ *approaches* 90° the tangent increases without any *fixed* limit." (The italics are the reviewer's). The mathematical concepts of "approaching" and "limit"—fixed or otherwise—though simple, are deep enough to justify avoidance in an elementary *Trigonometry;* or if included must be given their precise meaning. These facts should have been observed according to the authors themselves (§81, p. 69). There is no room in mathematics for "strictly" or "vaguely" speaking; there is only room for "correctly" speaking.

It would be a little embarrassing to answer a reasonable student's query: "On p. 95 we have the formula $\tan \alpha = -\tan(180° - \alpha)$ which will be proved to hold in general later. Now put $\alpha = 90°$; then $\tan 90° = -\tan 90°$ or $2 \tan 90° = 0$ and so $\tan 90° = 0$.

Would it not be much simpler to omit "strictly speaking" from the first of the above quotations and to say that $\tan \alpha = -\tan(180° - \alpha)$ is an identity in the sense that it is true for all values of α for which the symbols have a meaning? (See p. 103.)

All this however may be a matter of taste and the reviewer really holds the book in high esteem; he feels sure it will meet with much deserved success.

Washington State College. M. S. KNEBELMAN.